The foundational case against the ordinal-based proof of Goodstein's Theorem

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Abstract. For any given natural number m, Goodstein's Sequence $G(m)$ is a well-defined sequence of natural numbers greater than 0. The ordinal-based proof of Goodstein's Theorem is the argument that the sequence is necessarily finite since, first, it can be put in a putative 1-1 correspondence with a strictly descending (as defined with respect to the membership-based relation '$>_o$' between ordinals) sequence of ordinals below ϵ_0; and second there can be no infinitely descending sequence of ordinals. However, we show first that $G(m)$ is a finite sequence if, and only if, it can be put in a 1-1 correspondence with a strictly descending (as defined with respect to the inequality relation '$>$' between natural numbers) sequence of natural numbers; and second that $G(m)$ can be put in a 1-1 correspondence—in Goodstein's sense—with a strictly descending sequence of ordinals below ϵ_0 even if it *cannot* be put in a 1-1 correspondence with a strictly descending sequence of natural numbers. We conclude that, from a foundational perspective, Goodstein's ordinal-based argument does not support the conclusion that every Goodstein sequence over the natural numbers is finite. We further show that a PA-formula such as $[P(x)]$ cannot be interpreted as '$x \in \omega$ & $P^*(x)$' in any model of the first order Peano Arithmetic PA, where $P^*(x)$ is the interpretation of $[P(x)]$ under the standard interpretation of PA and 'ω' is the first limit ordinal. We conclude that this is a curious case of *validly* proving a Theorem involving the ordinal membership relation '$>_o$' over the structure of the ordinals below ϵ_0 and—ignoring the issue raised by Skolem's paradox—*invalidly* postulating that it *must* therefore hold for the natural number inequality relation '$>$' over the structure of the natural numbers since the natural numbers can be meta-mathematically presumed to be capable of being put in a putative 1-1 correspondence with the finite ordinals.

Keywords. First order Peano Arithmetic PA; Goodstein's sequence; hereditary representation; limit ordinals ω and ϵ_0; Löwenheim-Skolem Theorem; natural numbers; ordinal numbers; set theory ZF; Skolem's paradox; transfinite ordinals.

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Contents

1. Introduction

The set-theoretical argument for Goodstein's Theorem meets William Gasarch's criteria[1] of an argument that prima facie defies belief. We show that the disbelief is justified, since Goodstein's argument can be carried out completely over the structure $\mathbb{N}$ of the natural numbers without appealing to any properties of transfinite ordinal sequences.

However we cannot conclude from the arithmetical argument that every Goodstein sequence over the natural numbers[2] must terminate finitely.

We argue that Goodstein's argument is a curious case of proving a Theorem involving the set-theoretical membership-based relation '$>_o$' over the structure of the ordinals below ϵ_o and—ignoring Thoraf Skolem's cautionary remarks about unrestrictedly corresponding putative mathematical entities across domains of different axiom systems[3]—*invalidly* postulating that a corresponding theorem involving the natural number inequality relation '$>$' *must* therefore hold over the structure of the natural numbers.

1.A. Some properties of Goodstein's sequence over the natural numbers

We note that, for any natural number m, R. L. Goodstein[4] uses the properties of the hereditary representation of m to construct a sequence $G(m) \equiv \{g_1(m),\ g_2(m),\ \ldots\}$ of natural numbers by an unusual, but valid, algorithm[5].

> *Hereditary representation*: The representation of a number as a sum of powers of a base b, followed by expression of each of the exponents as a sum of powers of b, etc., until the process stops. For example, we may express the hereditary representations of 266 in base 2 and base 3 as follows:
>
> $$226_{[2]} \equiv 2^{8_{[2]}} + 2^{3_{[2]}} + 2 \equiv 2^{2^{(2^{2^0}+2^0)}} + 2^{2^{2^0}+2^0} + 2^{2^0}$$
>
> $$226_{[3]} \equiv 2.3^{4_{[3]}} + 2.3^{3_{[3]}} + 3^{2_{[3]}} + 1 \equiv 2.3^{(3^{3^0}+3^0)} + 2.3^{3^{3^0}} + 3^{2.3^0} + 3^0$$

For the moment we shall ignore the peculiar manner of constructing the individual members of the Goodstein sequence, since these are not germane to understanding the essence of Goodstein's argument. We need simply accept for now that $G(m)$ is well-defined over the structure $\mathbb{N}$ of the natural numbers, and has the following properties:

(i) For any given natural number $k > 0$ we can construct a hereditary representation—denoted by $g_k(m)_{[k+1]}$[6]—of $g_k(m)$ in the base $[k+1]$;

> *Example*: The hereditary representations of the first two terms of $G(226)$ are[7]:
>
> $$g_1(226)_{[2]} \equiv 2^{2^{2+1}} + 2^{2+1} + 2$$
>
> $$g_2(226)_{[3]} \equiv 3^{3^{3+1}} + 3^{3+1} + 2$$

(ii) We can define a Goodstein Functional Sequence $G(m)_{[x]} \equiv \{g_k(m)_{[(k+1)\ \hookrightarrow\ x]} : k > 0\}$ over $\mathbb{N}$ by replacing the base $[k+1]$ in $g_k(m)_{[k+1]}$ with the variable x for each $k > 0$;

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[1] [Ga10].
[2] Defined formally in Section 2.A..
[3] [Sk22].
[4] [Gd44].
[5] We define this formally in Section 2.A..
[6] Strictly speaking the denotation should be: $(g_k(m))_{[k+1]}$.
[7] For ease of expression, from now on we shall express 'a^0' as '1', and 'a^{b^0}' as 'a' unless indicated to the contrary.

Example:

$g_1(226)_{[2 \hookrightarrow x]}$[8] $\equiv x^{x^{x+1}} + x^{x+1} + x$

$g_2(226)_{[3 \hookrightarrow x]} \equiv x^{x^{x+1}} + x^{x+1} + 2$

(iii) We can show that some member of Goodstein's sequence $G(m)$ evaluates to 0 if, and only if, there is some natural number z such that for any given natural number $k > 0$:

If $g_k(m)_{[(k+1) \hookrightarrow z]} > 0$ in $G(m)_{[z]}$, then $g_k(m)_{[(k+1) \hookrightarrow z]} > g_{k+1}(m)_{[(k+2) \hookrightarrow z]}$.

Terminate finitely: By Goodstein's algorithm, after a 0 all subsequent members of the sequence necessarily remain 0, and the sequence is said in such a case to terminate finitely at its first 0 value.

The proof of (iii)—which depends, of course, on the peculiar nature of Goodstein's algorithm—is straightforward and detailed in Section 2.. The main point to note is that the proof appeals only to the usual properties of the natural numbers.

The question arises: Are we free to *postulate* the existence of such a natural number z, and conclude that some member of $G(m)$ must evaluate to 0 in $\mathbb{N}$?

Though it sounds absurd, the following theorem shows that this is precisely the freedom that the ordinal-based argument for Goodstein's Theorem (Section 2.D.) claims (albeit implicitly)!

Theorem 1.1. *Goodstein's ordinal sequence $G_o(m_o)$ over the finite ordinals terminates with respect to the ordinal inequality '$>_o$' in any sound interpretation of ZF even if Goodstein's natural number sequence $G(m)$ over the natural numbers does not terminate with respect to the natural number inequality '$>$' in any sound interpretation of PA.*

Proof: Assume that Goodstein's natural number sequence $G(m) \equiv \{g_k(m)_{[(k+1)} : k > 0\}$ over the natural numbers *does not terminate* with respect to the natural number inequality '$>$' in any sound interpretation of PA.

Let n_{max} be the largest term amongst the first n terms of $G(m)$. It is tedious but straightforward to show that, by our assumption, n_{max} is a monotonically increasing function of n. Hence there is *no* natural number z such that $g_k(m)_{[(k+1) \hookrightarrow z]} > g_{k+1}(m)_{[(k+2) \hookrightarrow z]}$ for all $k > 0$.

Consider next Goodstein's ordinal number sequence $G_o(m_o) \equiv \{g_k(m_o) : k > 0\}$ over the finite ordinals. There *is* now the axiomatically postulated transfinite ordinal ω such that $g_k(m_o)_{[(k+1) \hookrightarrow \omega]} >_o g_{k+1}(m_o)_{[(k+2) \hookrightarrow \omega]}$ for all $k > 0$.

Since there are no infinite descending sequences of ordinals with respect to the ordinal inequality '$>_o$', Goodstein's ordinal number sequence $G_o(m_o)$ must terminate finitely with respect to the ordinal inequality '$>_o$' in any sound interpretation of ZF. □

Since the finite ordinals can be meta-mathematically put into a 1-1 correspondence with the natural numbers, it follows that:

Corollary 1.2. *The relationship of terminating finitely with respect to the ordinal inequality '$>_o$' over an infinite set Z_0 of ordinals containing a transfinite ordinal in any sound interpretation of ZF cannot be corresponded to the relationship of terminating finitely with respect to the natural number inequality '$>$' over an infinite set N of natural numbers in any sound interpretation of PA.*

We now analyse Goodstein's argument in greater detail.

[8]We prefer this notation to that of the usual 'base bumping' function[9] as it makes the argument in Section 2.B. more transparent.

1.B. The argument of Goodstein's Theorem

The argument of Goodstein's Theorem[10] is that:

(i) The natural number considerations involved in the construction of Goodstein's sequence can all be formalised over the finite ordinals (sets) in any putative model $\mathbb{M}$ of ZF;

(ii) The comprehension axiom of ZF does allow us to *postulate* the existence of an ordinal—Cantor's first limit ordinal ω—such that:

(a) if $\{g_k(m_o)\}$[11]—say $G_o(m_o)$—is the sequence of finite ordinals in $\mathbb{M}$ that corresponds to Goodstein's natural number sequence $G(m)$ in $\mathbb{N}$,

(b) and $\{g_k(x)_{[(k_o+_o 1_o) \hookrightarrow x]} : k > 0\}$—say $G_o(m_o)_{[x]}$—the corresponding Goodstein Functional Sequence over $\mathbb{M}$,

(c) then for any given natural number $k > 0$:

If $g_k(m_o)_{[(k_o+_o 1_o) \hookrightarrow \omega]} >_o 0_o$ in $G_o(m_o)_{[\omega]}$, then
$g_k(m_o)_{[(k_o+_o 1_o) \hookrightarrow \omega]} >_o g_{k+1}(m_o)_{[(k_o+_o 2_o) \hookrightarrow \omega]}$;

(iii) The sequence $\{g_1(m_o)_{[2_o \hookrightarrow \omega]}, g_2(m_o)_{[3_o \hookrightarrow \omega]}, \ldots\}$ of ordinals cannot descend infinitely in $\mathbb{M}$;

(iv) Hence $G_o(m_o)$ terminates finitely in $\mathbb{M}$.

If ZF is consistent, then such a $\mathbb{M}$ must 'exist' and the above argument is valid in $\mathbb{M}$. However, Goodstein's Theorem is the conclusion that Goodstein's sequence must *therefore* terminate finitely in $\mathbb{N}$!

Prima facie such a conclusion from the ordinal-based reasoning challenges belief insofar as we shall show that—at heart—the argument *essentially* appears to be that, since Goodstein's natural number sequence $G(m)$ obviously 'terminates finitely' if, and only if, it is bounded above in $\mathbb{N}$[12] with respect to the arithmetical relation '$>$', we may conclude the existence of such a bound since Goodstein's ordinal sequence $G_o(m_o)$:

(a) *is* bounded above by ω in $\mathbb{M}$;

(b) 'terminates finitely' with respect to the ordinal relation '$>_o$';

(c) can be put in a 1-1 correspondence with $G(m)$;

and since the natural numbers can be put into a 1-1 correspondence with the finite ordinals!

We now show why such disbelief is justified since—as we detail in Section 3.—the above invalidly presumes that the structure $\mathbb{N}$ of the natural numbers is isomorphic to the sub-structure of the finite ordinals in the structure of the ordinals below ϵ_0, and so the property of 'terminating finitely' in any model of ZF must interpret as the property of 'terminatingly finitely' in any model of PA.

[10] Compare [Ca07].

[11] Notation: We denote by m_o the ordinal corresponding to the natural number m; by '$+_o$' and '$>_o$' the function/relation letters in ordinal arithmetic corresponding to the function/relation letters '$+$' and '$>$' in Peano Arithmetic, etc.

[12] Although we do not address the question here, it can be shown without appealing to any transfinite considerations that $G(m)$ cannot oscillate for any natural number m.

2. The case against the ordinal-based 'proof' of Goodstein's Theorem

For any given natural number m we can express $G(m)$ so that each term is expressed in it's hereditary representation:

$$G(m) \quad \equiv \quad \{g_1(m)_{[2]},\ g_2(m)_{[3]},\ g_3(m)_{[4]}, \ldots\} \tag{2.1}$$

where the first term $g_1(m)_{[2]}$ denotes the unique hereditary representation of the natural number m in the natural number base [2]:

e. g. $g_1(9)_{[2]} \equiv 1.2^{(1.2^{1.2^0}+1.2^0)} + 0.2^{(1.2^{1.2^0}+0.2^0)} + 0.2^{1.2^0} + 1.2^0$

and if $n > 1$ then $g_{(n)}(m)_{[n+1]}$ is defined recursively from $g_{(n-1)}(m)_{[n]}$ as below.

2.A. The recursive definition of Goodstein's Sequence

For $n > 1$ let the $(n-1)^{th}$ term $g_{(n-1)}(m)$ of the Goodstein sequence $G(m)$ be expressed syntactically by its hereditary representation as:

$$g_{(n-1)}(m)_{[n]} \quad \equiv \quad \sum_{i=0}^{l} a_i.n^{i_{[n]}} \tag{2.2}$$

where:

(a) $0 \leq a_i < n$ over $0 \leq i \leq l$;
(b) $a_l \neq 0$;
(c) for each $0 \leq i \leq l$ the exponent i too is expressed syntactically by its hereditary representation $i_{[n]}$ in the base $[n]$; as also are all of its exponents and, in turn, all of their exponents, etc.

We then define the n^{th} term of $G(m)$ as:

$$g_n(m) \quad = \quad \sum_{i=0}^{l}(a_i.(n+1)^{i_{[n \hookrightarrow (n+1)]}}) - 1 \tag{2.3}$$

2.A.a. The hereditary representation of $g_n(m)$

Now we note that:

(a) if $a_0 \neq 0$ then the hereditary representation of $g_n(m)$ is:

$$g_n(m)_{[n+1]} \quad \equiv \quad \sum_{i=1}^{l}(a_i.(n+1)^{i_{[n \hookrightarrow (n+1)]}}) + (a_0 - 1) \tag{2.4}$$

(b) whilst if $a_i = 0$ for all $0 \leq i < k$, then the hereditary representation of $g_n(m)$ is:

$$g_n(m)_{[n+1]} \quad \equiv \quad \sum_{i=k+1}^{l}(a_i.(n+1)^{i_{[n \hookrightarrow (n+1)]}}) + c_{k[n+1]} \tag{2.5}$$

where:

$$\begin{aligned} c_k &= a_k.(n+1)^{k_{[n \hookrightarrow (n+1)]}} - 1 \\ &= (a_k - 1).(n+1)^{k_{[n \hookrightarrow (n+1)]}} + \left\{(n+1)^{k_{[n \hookrightarrow (n+1)]}} - 1\right\} \\ &= (a_k - 1).(n+1)^{k_{[n \hookrightarrow (n+1)]}} + n\left\{(n+1)^{k_{[n \hookrightarrow (n+1)]}-1} + (n+1)^{k_{[n \hookrightarrow (n+1)]}-2} \ldots + 1\right\} \end{aligned}$$

and so its hereditary representation in the base $(n+1)$ is given by:

$$c_{k[n+1]} \equiv (a_k - 1).(n+1)^{k_{1[n+1]}} + n\left\{(n+1)^{k_{2[n+1]}} + (n+1)^{k_{3[n+1]}} \ldots + 1\right\}$$

where $k_{1[n+1]} \equiv k_{[n \hookrightarrow (n+1)]}$ and $k_1 > k_2 > k_3 > \ldots \geq 1$.

2.B. Goodstein's argument in arithmetic

For $n > 1$ we consider the difference:

$$d_{(n-1)} = \left\{g_{(n-1)}(m)_{[n]} - g_n(m)_{[n+1]}\right\}$$

Now:

(a) if $a_0 \neq 0$ we have:

$$d_{(n-1)} = \sum_{i=0}^{l}(a_i.n^{i_{[n]}}) - \sum_{i=1}^{l}(a_i.(n+1)^{i_{[n \hookrightarrow (n+1)]}}) - (a_0 - 1) \tag{2.6}$$

(b) whilst if $a_i = 0$ for all $0 \leq i < k$ we have:

$$\begin{aligned} d_{(n-1)} = & \sum_{i=k}^{l}(a_i.n^{i_{[n]}}) - \sum_{i=(k+1)}^{l}(a_i.(n+1)^{i_{[n \hookrightarrow (n+1)]}}) - \\ & (a_k - 1).(n+1)^{k_{1[n+1]}} - \\ & n\left\{(n+1)^{k_{2[n+1]}} + (n+1)^{k_{3[n+1]}} \ldots + 1\right\} \end{aligned} \tag{2.7}$$

Further:

(c) if in equation (2.6) we replace the base $[n]$ by the base $[z]$ in the term:

$$\sum_{i=0}^{l} a_i.n^{i_{[n]}} \tag{2.8}$$

and the base $[n+1]$ also by the base $[z]$ in the term:

$$\sum_{i=k+1}^{l}(a_i.(n+1)^{i_{[n \hookrightarrow (n+1)]}}) + (a_0 - 1) \tag{2.9}$$

then we have:

$$\begin{aligned} d'_{(n-1)} &= \sum_{i=0}^{l}(a_i.z^{i_{[n \hookrightarrow z]}}) - \sum_{i=1}^{l}(a_i.z^{i_{[n \hookrightarrow z]}}) - (a_0 - 1) \\ &= 1 \end{aligned} \tag{2.10}$$

since $(i_{[n \hookrightarrow (n+1)]})_{[(n+1) \hookrightarrow z]} \equiv i_{[n \hookrightarrow z]}$;

(d) whilst if in equation (2.7) we replace the bases similarly, then we have:

$$\begin{aligned} d'_{(n-1)} &= \sum_{i=k}^{l}(a_i.z^{i_{[n \hookrightarrow z]}}) - \sum_{i=(k+1)}^{l}(a_i.z^{i_{[n \hookrightarrow z]}}) - \\ & \quad (a_k - 1).z^{k_{1[(n+1) \hookrightarrow z]}} - n\left\{z^{k_{2[(n+1) \hookrightarrow z]}} + z^{k_{3[(n+1) \hookrightarrow z]}} \ldots + 1\right\} \\ &= a_k.z^{k_{[n \hookrightarrow z]}} - (a_k - 1).z^{k_{1[(n+1) \hookrightarrow z]}}) - n(z^{k_{2[(n+1) \hookrightarrow z]}} + z^{k_{3[(n+1) \hookrightarrow z]}} \ldots + 1) \\ &= z^{k_{1[(n+1) \hookrightarrow z]}} - n(z^{k_{2[(n+1) \hookrightarrow z]}} + z^{k_{3[(n+1) \hookrightarrow z]}} \ldots + 1) \end{aligned} \tag{2.11}$$

where $k_{1[(n+1) \hookrightarrow z]} \equiv k_{[n \hookrightarrow z]}$, and $k_{1[(n+1) \hookrightarrow z]} > k_{2[(n+1) \hookrightarrow z]} > k_{3[(n+1) \hookrightarrow z]} > \ldots \geq 1$.

We consider now the sequence:

$$G(m)_{[z]} \equiv (g_1(m)_{[2 \hookrightarrow z]},\ g_2(m)_{[3 \hookrightarrow z]},\ g_3(m)_{[4 \hookrightarrow z]}, \ldots)$$

obtained from Goodstein's sequence by replacing the base $[n+1]$ in each of the terms $g_n(m)_{[n+1]}$ by the base $[z]$ for all $n \geq 1$.

Clearly if $z > n$ for all non-zero terms of the Goodstein sequence, then $d'_{(n-1)} > 0$ in each of the cases—equation (2.10) and equation (2.11).

Since we then have $d'_{(n-1)} \geq (z^k - (z-1)(z^{(k-1)} + z^{(k-2)} + z^{(k-3)} + \ldots + 1)) = 1$ in equation (2.11).

The sequence $G(m)_{[z]}$ is then a descending sequence of natural numbers, and must terminate finitely in $\mathbb{N}$ if $z > n$.

Since $g_n(m)_{[(n+1) \hookrightarrow z]} \geq g_n(m)_{[n+1]}$ if $z > n$, Goodstein's sequence $G(m)$ too must terminate finitely in $\mathbb{N}$ if $z > n$.

Obviously, since we can always find a $z > n$ for all non-zero terms of the Goodstein sequence if it terminates finitely in $\mathbb{N}$, the condition that we can always find some $z > n$ for all non-zero terms of any Goodstein sequence is equivalent to the assumption that any Goodstein sequence terminates finitely in $\mathbb{N}$.

2.C. Goodstein's argument in set theory

Now the set-theoretical form of the argument due to Goodstein is essentially that:

(a) if we take the value of x in the Goodstein Functional Sequence $G_o(m_o)_{[x]}$ over the finite ordinals to be the first limit ordinal ω,

(b) and consider the—necessarily decreasing in this case—ordinal sequence (corresponding to the conditionally decreasing natural number sequence $G(m)_{[z]}$):

$$G_o(m_o)_{[\omega]} \equiv \{g_1(m_o)_{[2_o \hookrightarrow \omega]},\ g_2(m_o)_{[3_o \hookrightarrow \omega]},\ g_3(m_o)_{[4_o \hookrightarrow \omega]}, \ldots\}$$

(c) then—since the ordinal numbers are well-ordered, and contain a subset of ω that can be put in a 1-1 correspondence with the set of natural numbers—we may conclude that $\omega > n$ for any given natural number n in some putative non-standard model of first order PA;

(d) hence we need not bother to establish a proof that some natural number $z > n$, too, always exists for all non-zero terms of any Goodstein sequence over the natural numbers in the model,

(e) and, since $G(m)$ and $G_o(m_o)_{[\omega]}$ can always be put in a 1-1 correspondence meta-mathematically, we may meta-mathematically conclude that every Goodstein sequence over the natural numbers terminates finitely over the structure $\mathbb{N}$ of the natural numbers.

However we note that if there is no natural number z such that $z > n$ for all non-zero terms of some Goodstein sequence, then:

(i) For *any* given n, we can find a z such that the the first n terms of the sequence $G(m)_{[z]}$ are a descending sequence of natural numbers in $\mathbb{N}$;

(ii) The sequence $G_o(m_o)_{[\omega]}$ is a finite descending sequence of ordinal numbers in $\mathbb{M}$

The ordinal-based proof of Goodstein's Theorem is thus the postulation that $G(m)_{[z]}$ and $G_o(m_o)_{[\omega]}$ can always be put in a 1-1 correspondence, and so the above is a contradiction from which we may conclude that there is always some natural number z such that $z > n$ for all non-zero terms of the Goodstein sequence $G(m)$!

Such a conclusion, however, ignores the cautionary remarks by Thoraf Skolem—about unrestrictedly corresponding meta-mathematically putative mathematical entities across domains of different axiom systems—in his 1922 address delivered in Helsinki before the Fifth Congress of Scandinavian Mathematicians, where Skolem improved upon both the argument and statement of Löwenheim's 1915 theorem[13]—subsequently labelled as the (downwards) Löwenheim-Skolem Theorem[14].

> **(Downwards) Löwenheim-Skolem Theorem**[15]: If a first-order proposition is satisfied in any domain at all, then it is already satisfied in a denumerably infinite domain.

Skolem then drew attention to a [16]:

> **Skolem's (apparent) paradox**: "...peculiar and apparently paradoxical state of affairs. By virtue of the axioms we can prove the existence of higher cardinalities, of higher number classes, and so forth. How can it be, then, that the entire domain B can already be enumerated by means of the finite positive integers? The explanation is not difficult to find. In the axiomatization, "set" does not mean an arbitrarily defined collection; the sets are nothing but objects that are connected with one another through certain relations expressed by the axioms. Hence there is no contradiction at all if a set M of the domain B is non-denumerable in the sense of the axiomatization; for this means merely that *within* B there occurs no one-to-one mapping Φ of M onto Z_o (Zermelo's number sequence). Nevertheless there exists the possibility of numbering all objects in B , and therefore also the elements of M, by means of the positive integers; of course such an enumeration too is a collection of certain pairs, but this collection is not a "set" (that is, it does not occur in the domain B)."

2.D. Goodstein's Theorem

Formally, Goodstein's ordinal-based argument is that since there are no infinitely descending sequences of ordinals, the sequence of ordinal numbers:

$$G_o(m_o)_{[\omega]} \equiv \{g_1(m_o)_{[2_o \hookrightarrow \omega]},\ g_2(m_o)_{[3_o \hookrightarrow \omega]},\ g_3(m_o)_{[4_o \hookrightarrow \omega]}, \ldots\}$$

can be shown to terminate finitely for any given finite ordinal m_o in any model $\mathbb{M}$ of ZF.

Hence the following proposition—where $g_y(x)$ denotes the y^{th} term of the Goodstein ordinal sequence $G_o(x)$—would hold in every model of ZF:

$$(\forall x)((x \in \omega) \rightarrow (\exists y)((y \in \omega) \wedge g_y(x) = 0_o))$$

Goodstein's Theorem over the natural numbers is then the conclusion that:

$$(\exists y)(g_y(m)) = 0$$

holds for any given natural number m in the standard interpretation of the first order Peano Arithmetic PA.

However this argument implicitly assumes that every putative model of ZF is a model of PA.

[13] [Lo15], p.235, Theorem 2.
[14] [Sk22], p.293.
[15] [Lo15], p.245, Theorem 6; [Sk22], p.293.
[16] [Sk22], p.295.

2.E. The case against unrestrictedly concluding arithmetical properties from set-theoretical reasoning

Now there *is* a suitable interpretation of the primitive symbols of PA in ZF which *transforms* the axioms of PA into theorems of ZF whilst preserving its rules of inference[17].

This interpretation *explicitly* assumes the existence of Cantor's first limit-ordinal ω in the domain of the interpretation so that the PA-formula $[(\forall x)(f(x))]$, for instance, *transforms* as $[(\forall x)((x \in \omega) \rightarrow (f(x)))]$.

Every model of ZF is then a model of the *transformed* axioms of PA.

The question is: Can we assume that some model of ZF is also a model of PA?

Now we show in Section 3. that this is impossible since, in any model of first-order PA, every element in the domain of the interpretation—except 0—is necessarily a successor[18].

Hence *no* model of PA can have an initial non-successor limit-ordinal ω such that $\omega > n$ for all natural numbers $n \geq 1$.

Thus Goodstein's argument arguably conflates a postulated interpretation of PA—which *transforms* some formulas of PA into provable formulas of ZF—with an interpretation of PA under which these formulas are *held* to be true over $\mathbb{N}$.

3. Does PA *really* have a non-standard model?

We now highlight the need to distinguish between the essentially different roles and requirements of mathematical languages of rich and adequate expression (such as ZF) vis-à-vis those of effective and unambiguous communication (such as PA)—whose appreciation may be significant for determining logical limits on the interpretation of number-theory in set-theory.

A particular limitation is reflected in the following argument that PA does not admit a non-standard[19] model of PA[20].

This challenges the impression given by informal arguments in standard texts[21], which suggest that we may trivially infer the existence of non-standard models of PA from standard set-theoretic theorems of first-order logic.

Another limitation is reflected in the argument that although every PA-theorem relativises to a corresponding ZF-theorem that holds over the finite ordinals *if* ZF is consistent and has a model, we cannot presume that if a ZF-theorem holds over the finite ordinals in a putative model, then it must yield a corresponding PA-theorem that holds over the natural numbers similarly.

We show below that such unqualified extension of set-theoretic reasoning to number-theory can be both misleading and invalid.

3.A. Why PA *cannot* admit a set-theoretical model

Let $[G(x)]$ denote the PA-formula:

$[x = 0 \vee \neg(\forall y)\neg(x = y')]$

This translates, under every unrelativised interpretation of PA, as:

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[17] [Me64], p192.

[18] See also [An08].

[19] Cf. [Me64], p107; [Sc67], p23, p209; [BBJ03], p104 for formal definitions of the standard interpretation/model of PA.

[20] Hence PA has no non-standard models under any sound interpretation; see [An14].

[21] eg. [BBJ03]. p306; [Me64], p112, Ex. 2.

If x denotes an element in the domain of an unrelativised interpretation of PA, either x is 0, or x is a 'successor'.

Further, in every such interpretation of PA, if $G(x)$ denotes the interpretation of $[G(x)]$:

(a) $G(0)$ is true;
(b) If $G(x)$ is true, then $G(x')$ is true.

Hence, by Gödel's completeness theorem:

(c) PA proves $[G(0)]$;
(d) PA proves $[G(x) \rightarrow G(x')]$.

Gödel's Completeness Theorem: In any first-order predicate calculus, the theorems are precisely the logically valid well-formed formulas (*i. e. those that are true in every model of the calculus*).

Further, by Generalisation:

(e) PA proves $[(\forall x)(G(x) \rightarrow G(x'))]$;

Generalisation in PA: $[(\forall x)A]$ follows from $[A]$.

Hence, by Induction:

(f) $[(\forall x)G(x)]$ is provable in PA.

Induction Axiom Schema of PA: For any formula $[F(x)]$ of PA:
$[F(0) \rightarrow ((\forall x)(F(x) \rightarrow F(x')) \rightarrow (\forall x)F(x))]$

In other words, except 0, every element in the domain of any unrelativised interpretation of PA is a 'successor'. Further, x can only be a 'successor' of a unique element in any such interpretation of PA.

3.A.a. PA and ZF have no common model

Now, since Cantor's first limit ordinal, ω, is not the 'successor' of any ordinal in the sense required by the PA axioms, and if there are no infinitely descending sequences of ordinals[22] in a model—if any—of set-theory, PA and Ordinal Arithmetic[23] cannot have a common model, and so we cannot consistently extend PA to ZF simply by the addition of more axioms.

3.A.b. Why the usual argument *for* a non-standard model of PA is unconvincing

Further, although we *can* define a model of Arithmetic with an infinite descending sequence of elements[24], any such model is isomorphic to the "*true arithmetic*"[25] of the integers (*negative plus positive*), and *not* to any model of PA[26].

Moreover—as we show in the next section—we cannot assume that we can consistently add a constant c to PA, along with the denumerable axioms $[\neg(c = 0)]$, $[\neg(c = 1)]$, $[\neg(c = 2)]$, $\ldots$, since this would presume that which is sought to be proven, viz., that PA has a non-standard model.

We *cannot* therefore—as suggested in standard texts[27]—apply the Compactness Theorem and the (*upward*) Löwenheim-Skolem Theorem to conclude that PA has a non-standard model!

Compactness Theorem: If every finite subset of a set of sentences has a model, then the whole set has a model[28].

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[22] cf. [Me64], p261.
[23] cf. [Me64], p.187.
[24] eg. [BBJ03], Section 25. 1, p303.
[25] [BBJ03]. p150. Ex. 12. 9.
[26] [BBJ03]. Corollary 25. 3, p306.
[27] eg. [BBJ03]. p306; [Me64], p112, Ex. 2.
[28] [BBJ03]. p147.

Upward Löwenheim-Skolem Theorem: Any set of sentences that has an infinite model has a non-denumerable model[29].

3.B. A formal argument *for* a non-standard model of PA

The following argument[30] attempts to validate the above line of reasoning suggested by standard texts *for* the existence of non-standard models of PA:

1. Let $<N$ (*the set of natural numbers*); $=$ (*equality*); $'$ (*the successor function*); $+$ (*the addition function*); $*$ (*the product function*); 0 (*the null element*)$>$ be the structure that serves to define a *sound* interpretation of PA, say $[N]$.
2. Let $\mathrm{T}[N]$ be the set of PA-formulas that are satisfied or true in $[N]$.
3. The PA-provable formulas form a subset of $\mathrm{T}[N]$.
4. Let Γ be the countable set of all PA-formulas of the form $[c_n = (c_{n+1})']$, where the index n is a natural number.
5. Let T be the union of Γ and $\mathrm{T}[N]$.
6. $\mathrm{T}[N]$ plus any finite set of members of Γ has a model, e.g., $[N]$ itself, since $[N]$ is a model of any finite descending chain of successors.
7. Consequently, by Compactness, T has a model; call it M.
8. M has an infinite descending sequence with respect to $'$ because it is a model of Γ.
9. Since PA is a subset of T, M is a non-standard model of PA.

Now, if—as claimed above—$[N]$ is a model of $\mathrm{T}[N]$ plus any finite set of members of Γ, then all PA-formulas of the form $[c_n = (c_{n+1})']$ are PA-provable, Γ is a proper sub-set of the PA-provable formulas, and T is identically $\mathrm{T}[N]$.

Reason: The argument cannot be that some PA-formula of the form $[c_n = (c_{n+1})']$ is true in $[N]$, but not PA-provable, as this would imply that PA+$[\neg(c_n = (c_{n+1})')]$ has a model other than $[N]$; in other words, it would presume that PA has a non-standard model.[31]

Consequently, the postulated model M of T in (7), by "Compactness", is the model $[N]$ that defines $\mathrm{T}[N]$. However, $[N]$ has no infinite descending sequence with respect to $'$, even though it is a model of Γ. Hence the argument does not establish the existence of a non-standard model of PA with an infinite descending sequence with respect to the successor function $'$.

3.C. The (*upward*) Skolem-Löwenheim theorem applies only to first-order theories that admit an axiom of infinity

We note, moreover, that the non-existence of non-standard models of PA would not contradict the (*upward*) Skolem-Löwenheim theorem, since the proof of this theorem implicitly limits its applicability amongst first-order theories to those that are consistent with an axiom of infinity—in the sense that

[29] [BBJ03]. p163.

[30] [Lu08].

[31] The same objection applies to the usual argument found in standard texts (eg.[?, BBJ03] p306; [Me64], p112, Ex. 2; [Ka91]; [Ka11]; see also [An14]) which, again, is essentially that, if PA has a non-standard model at all, then one such model is obtained by assuming we can consistently add a single non-numeral constant c to the language of PA, and the countable axioms $c \neq 0$, $c \neq 1$, $c \neq 2$, ... to PA. However, as noted earlier, this argument too does not resolve the question of whether such assumption validly allows us to conclude that there is a non-standard model of PA in the first place.

To place this distinction in perspective, Legendre and Gauss independently conjectured in 1796 that, if $\pi(x)$ denotes the number of primes less than x, then $\pi(x)$ is asymptotically equivalent to $x/\mathrm{In}(x)$. Between 1848/1850, Chebyshev confirmed that if $\pi(x)/\{x/\mathrm{In}(x)\}$ has a limit, then it must be 1. However, the crucial question of whether $\pi(x)/\{x/\mathrm{In}(x)\}$ has a limit at all was answered in the affirmative independently by Hadamard and de la Vallée Poussin only in 1896.

the proof implicitly requires that a constant, say c, along with a denumerable set of axioms to the effect that $c \neq 0, c \neq 1, \ldots$, can be consistently added to the theory. However, as seen in the previous section, this is not the case with PA.

3.D. *Why* PA has no set-theoretical model

We can define the usual order relation '$<$' in PA so that every instance of the Induction Axiom schema, such as, say:

(i) $[F(0) \rightarrow ((\forall x)(F(x) \rightarrow F(x')) \rightarrow (\forall x)F(x))]$

yields the PA theorem:

(ii) $[F(0) \rightarrow ((\forall x)((\forall y)(y < x \rightarrow F(y)) \rightarrow F(x)) \rightarrow (\forall x)F(x))]$

Now, if we interpret PA without relativisation in ZF in the sense indicated by Feferman[32] — i.e., numerals as finite ordinals, $[x']$ as $[x \cup \{x\}]$, etc. — then (ii) always translates in ZF as a theorem:

(iii) $[F(0) \rightarrow ((\forall x)((\forall y)(y \in x \rightarrow F(y)) \rightarrow F(x)) \rightarrow (\forall x)F(x))]$

However, (i) does not always translate similarly as a ZF-theorem (*which is why PA and ZF can have no common model*), since the following is not necessarily provable in ZF:

(iv) $[F(0) \rightarrow ((\forall x)(F(x) \rightarrow F(x \cup \{x\})) \rightarrow (\forall x)F(x))]$

Example: Define $[F(x)]$ as '$[x \in \omega]$'.

A significant point which emerges from the above is that we cannot appeal unrestrictedly to set-theoretical reasoning when studying the foundational framework of PA.

Reason: The language of PA has no constant that interprets in any model of PA as the set N of all natural numbers.

Moreover, the preceding sections show that the Induction Axiom Schema of PA does not allow us to bypass this constraint by introducing an "actual" (*or "completed"*) infinity disguised as an arbitrary constant - usually denoted by c or ∞ - into either the language, or a putative model, of PA.

4. The standard Gödelian representation of Goodstein's Theorem is not provable in PA

Since $g_y(x)$ is recursive—where $g_y(x)$ denotes the y^{th} term of the Goodstein ordinal sequence $G(x)$—by a standard representation theorem[33] due to Kurt Gödel[34], $g_y(x)$ is representable in PA by some formula $[G(x, y, z)]$. One formalisation of Goodstein's Theorem in PA would then be the assertion that PA proves $[(\forall x)(\exists y)\ (G(x, y, 0))]$. However any such proof would yield an algorithm that, for any natural number m, would provide evidence that $G^*(m, n, 0)$ holds for some natural number n, where $G^*(x, y, z)$ is the interpretation of $[G(x, y, z)]$ under a sound interpretation of PA over $\mathbb{N}$. Since it can be shown[35] that there can be no such algorithm we conclude that this formalisation of Goodstein's Theorem cannot be proven in PA even if every Goodstein sequence were to terminate finitely over $\mathbb{N}$.

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