Why Hilbert's and Brouwer's interpretations of quantification are complementary and not contradictory

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Abstract. We show that both Hilbert's and Brouwer's interpretations of quantification yield interpretations of the first-order Peano Arithmetic PA—over the structure $\mathbb{N}$ of the natural numbers—that are complementary, not contradictory. The former yields the standard interpretation $\boldsymbol{M}$ of PA over $\mathbb{N}$, which is *relatively* sound with respect to assignments of algorithmically verifiable Tarskian truth values to the formulas of PA under $\boldsymbol{M}$; and which can be viewed as circumscribing the ambit of *non-finitary* human reasoning about 'true' arithmetical propositions. The latter yields a finitary interpretation $\boldsymbol{B}$ of PA over $\mathbb{N}$, which is *relatively* sound with respect to assignments of algorithmically computable Tarskian truth values to the formulas of PA under $\boldsymbol{B}$; and which can be viewed as circumscribing the ambit of *finitary* mechanistic reasoning about 'true' arithmetical propositions.

Keywords. Peano Arithmetic, algorithmic computability, algorithmic verifiability, finitary, interpretation, model, soundness, truth assignments, ω-consistency.

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Contents

1. Introduction

We shall argue the thesis that the perceived conflict between David Hilbert's and L. E. J. Brouwer's interpretation of quantification is illusory, and merely reflects an unrecognised ambiguity whose roots trace back to antiquity in the non-finitary postulation:

- of an 'unspecified' element
- in Aristotle's logic of predicates.

This is the postulation that:

- If it is not the case that, for any specified x, $F(x)$ does not hold,
- then there exists an unspecified x, such that $F(x)$ holds.

Where 'holds' is to be understood in Tarski's sense that:

- 'Snow is white' holds as a true assertion if, and only if,
- it can be objectively determined, on the basis of evidence, that snow *is* white.

We shall show that recognition, and removal, of the ambiguity has significant consequences for the current perception that:

- Gödel's Incompleteness Theorems limit the effective assignments of truth values to the formulas of a mathematical language such as the first-order Peano Arithmetic PA.

Formally, we shall show that both Hilbert's and Brouwer's interpretations of quantification yield interpretations of the first-order Peano Arithmetic PA—over the structure $\mathbb{N}$ of the natural numbers—that are complementary, not contradictory.

The former yields the *standard* interpretation $\boldsymbol{M}$ of PA over $\mathbb{N}$, which is *relatively* sound with respect to *non-finitary* assignments of algorithmically verifiable Tarskian truth values $T_{\mathcal{M}}$ to the formulas of PA under $\boldsymbol{M}$; and which can be viewed as circumscribing the ambit of *non-finitary* human reasoning about 'true' arithmetical propositions.

The latter yields a *finitary* interpretation $\boldsymbol{B}$ of PA over $\mathbb{N}$, which is *relatively* sound with respect to *finitary* assignments of algorithmically computable Tarskian truth values $T_{\mathcal{B}}$ to the formulas of PA under $\boldsymbol{B}$; and which can be viewed as circumscribing the ambit of *finitary* mechanistic reasoning about 'true' arithmetical propositions.

Definition 1. *An interpretation $\mathcal{I}$ of a formal language L, over a domain D of a structure $\mathcal{S}$, is sound relative to an assignment of truth values $T_{\mathcal{I}}$ to the formulas of L if, and only if, the axioms of L interpret as true, and the rules of inference of L preserve truth, over D under $\mathcal{I}$ relative to the assignment of truth values $T_{\mathcal{I}}$.*

1.A. Hilbert's interpretation of quantification

Now, Hilbert defined a formal logic L_{ε}, in which he sought to capture the essence:

- of Aristotle's 'unspecified' x,
- as an 'unspecified' term $[\varepsilon_x(F(x))]$.

Hilbert then defined:

- $[(\forall x)F(x) \leftrightarrow F(\varepsilon_x(\neg F(x)))]$
- $[(\exists x)F(x) \leftrightarrow F(\varepsilon_x(F(x)))]$

and showed that Aristotle's logic is a sound interpretation of L_ε:

- if $[\varepsilon_x(F(x))]$ can be interpreted as some, 'unspecified', x satisfying $F(x)$.

Formally, Hilbert interpreted quantification in terms of his ε-function as follows[1]:

> "IV. The logical ε-axiom
>
> 13. $A(a) \rightarrow A(\varepsilon(A))$
>
> Here $\varepsilon(A)$ stands for an object of which the proposition $A(a)$ certainly holds if it holds of any object at all; let us call ε the logical ε-function.
>
> 1. By means of ε, "all" and "there exists" can be defined, namely, as follows:
>
> (i) $(\forall a)A(a) \leftrightarrow A(\varepsilon(\neg A))$
>
> (ii) $(\exists a)A(a) \leftrightarrow A(\varepsilon(A)) \ldots$
>
> On the basis of this definition the ε-axiom IV(13) yields the logical relations that hold for the universal and the existential quantifier, such as:
>
> $(\forall a)A(a) \rightarrow A(b)$ ...(Aristotle's dictum),
>
> and:
>
> $\neg((\forall a)A(a)) \rightarrow (\exists a)(\neg A(a))$ ...(principle of excluded middle)."

Thus, Hilbert's interpretation (i) of universal quantification—under any objective method T_H of assigning truth values to the sentences of a fornal logic L—is that the sentence $(\forall x)F(x)$ can be defined as holding (*under a consistent interpretation* $\boldsymbol{H}$ *of* L *with respect to* T_H) if, and only if, $F(a)$ holds whenever $\neg F(a)$ holds for some unspecified a (*in* $\boldsymbol{H}$); which would imply that $\neg F(a)$ does not hold for any specified a (*since* $\boldsymbol{H}$ *is consistent*), and so $F(a)$ holds for any specified a (*in* $\boldsymbol{H}$).

Further, Hilbert's interpretation (ii) of existential quantification, with respect to T_H, postulates that $(\exists x)F(x)$ holds (*in* $\boldsymbol{H}$) if, and only if, $F(a)$ holds for some unspecified a (*in* $\boldsymbol{H}$).

1.B. Brouwer's objection

Brouwer's objection to such an 'unspecified' and 'postulated' interpretation of quantification was that, for an interpretation to be considered *sound* relative to T_H when the domain of the quantifiers under an interpretation is infinite, the decidability of the quantification under the interpretation must be constructively verifiable in some intuitively and mathematically acceptable sense of the term 'constructive'[2].

Two questions arise:

(a) Is Brouwer's objection relevant today?
(b) If so, can we interpret quantification finitarily?

[1] [Hi27].
[2] [Br08].

2. *Is* Brouwer's objection relevant today?

2.A. Is the PA-formula $[(\forall x)F(x)]$ to be interpreted constructively or finitarily?

The perspective we choose for addressing these issues is that of the structure $\mathbb{N}$, defined by:

- $\{N$ (*the set of natural numbers*);
- $=$ (*equality*);
- S (*the successor function*);
- $+$ (*the addition function*);
- $*$ (*the product function*);
- 0 (*the null element*)$\}$

which serves for a definition of today's standard interpretation, say $\boldsymbol{M}$, of the first-order Peano Arithmetic PA.

However, if we are to avoid intuitionistic objections to the admitting of 'unspecified' natural numbers in the definition of quantification under $\boldsymbol{M}$, we are faced with the ambiguity where:

- if $[(\forall x)F(x)]$ and $[(\exists x)F(x)]$ denote PA-formulas,
- and the relation $F^*(x)$ denotes the interpretation in the standard interpretation $\boldsymbol{M}$ of the PA-formula $[F(x)]$ under an inductive assignment of Tarskian truth values T_M,
- then, in the underlying first-order logic FOL of PA that favours evidence-based[3] interpretation[4],

the question arises:

(a) Is the PA-formula $[(\forall x)F(x)]$ to be interpreted constructively as:

- '*For any n, $F^*(n)$*',
- which holds if, and only if,
- for *any specified* n in $\mathbb{N}$,
- there is *algorithmic evidence* that $F^*(n)$ holds in $\mathbb{N}$,

or:

(b) is the formula $[(\forall x)F(x)]$ to be interpreted finitarily as:

- '*For all n, $F^*(n)$*',
- which holds if, and only if,
- there is *algorithmic evidence* that,
- for *any specified* n in $\mathbb{N}$,
- $F^*(n)$ holds in $\mathbb{N}$?

where:

Definition 2. *A natural number n is defined as specifiable in $\mathbb{N}$ if, and only if, it can be explicitly denoted as a PA-numeral by a PA-formula that interprets as an algorithmically computable[5] constant.*

[3] cf. [Mu91]: "It is by now folklore . . . that one can view the *values* of a simple functional language as specifying *evidence* for propositions in a constructive logic . . . ".

[4] As introduced in [An12] and detailed below.

[5] i.e., 'algorithmically computable' as detailed in Definition 4.

2.B. The standard interpretation $\boldsymbol{M}$ of PA interprets $[(\forall x)F(x)]$ constructively

Keeping the above distinction in mind, it would seem that:

(1a) The formula $[(\forall x)F(x)]$ is *defined* as true in $\boldsymbol{M}$ relative to T_M if, and only if, for any specified natural number n, we may conclude on the basis of evidence-based reasoning that the proposition $F^*(n)$ holds in $\boldsymbol{M}$;

(1b) The formula $[(\exists x)F(x)]$ is an abbreviation of $[\neg(\forall x)\neg F(x)]$, and is *defined* as true in $\boldsymbol{M}$ relative to T_M if, and only if, it is not the case that, for any specified natural number n, we may conclude on the basis of evidence-based reasoning that the proposition $\neg F^*(n)$ holds in $\boldsymbol{M}$;

(1c) The proposition $F^*(n)$ is *postulated* as holding in $\boldsymbol{M}$ for some specified natural number n if, and only if, it is not the case that, for any specified natural number n, we may conclude on the basis of evidence-based reasoning that the proposition $\neg F^*(n)$ holds in $\boldsymbol{M}$.

If so, then (1a), (1b) and (1c) together interpret $[(\forall x)F(x)]$ and $[(\exists x)F(x)]$ under $\boldsymbol{M}$ as intended by Hilbert's ε-function; whence they attract Brouwer's objection.

This would, then, answer question (a).

3. *Can* we interpret quantification finitarily?

3.A. A finitary interpretation $\boldsymbol{B}$ of PA which interprets $[(\forall x)F(x)]$ finitarily

Now, our thesis is that the implicit target of Brouwer's objection[6] is the unqualified semantic postulation of Aristotle's particularisation (1c)[7], which appeals to Platonically non-constructive, rather than intuitively constructive, plausibility.

> We note that this conclusion about Brouwer's essential objection apparently differs from conventional intuitionistic wisdoms[8]:
>
> - which would presumably deny appeal to (1c) in an interpretation of FOL by denying the FOL theorem $[P\ v\ \neg P]$ (*Law of the Excluded Middle*);
> - even though denying appeal to (1c) in an interpretation of FOL does *not* entail denying the FOL theorem $[P\ v\ \neg P]$ (*Law of the Excluded Middle*).

We can thus re-phrase question (b) more specifically:

- Can we define an interpretation of PA over $\mathbb{N}$ that does not appeal to (1c)?

[6] And perhaps of parallel objections perceived generically as "*Limitations of first-order logic*"; [AR02b], p.78, §2.1.

[7] Although Brouwer's explicitly stated objection appeared to be to the Law of the Excluded Middle as expressed and interpreted at the time ([Br23], p.335-336; [Kl52], p.47; [Hi27], p.475), some of Kleene's remarks ([Kl52], p.49), some of Hilbert's remarks (for instance in [Hi27], p.474) and, more particularly, some of Kolmogorov's remarks (in [Ko25], fn. p.419; p.432), suggest that the intent of Brouwer's fundamental objection can also be viewed today as being limited only to the yet prevailing belief—as an article of faith—that the validity of Aristotle's particularisation can be extended without qualification to infinite domains, as detailed in §12..

[8] i.e., those based essentially on Brouwer's explicitly stated objection to the Law of the Excluded Middle as expressed in [Br23], p.335-336.

We shall now argue that[9] we *can*, indeed, define another—hitherto unsuspected—evidence-based interpretation $\boldsymbol{B}$ of PA under an inductive assignment of Tarskian truth values T_B over the structure $\mathbb{N}$, where:

(2a) The formula $[(\forall x)F(x)]$ is *defined* as true in $\boldsymbol{B}$ relative to T_B if, and only if, we may conclude on the basis of evidence-based reasoning that, for any specified natural number n, the proposition $F^*(n)$ holds in $\boldsymbol{B}$;

(2b) The formula $[(\exists x)F(x)]$ is an abbreviation of $[\neg(\forall x)\neg F(x)]$, and is *defined* as true in $\boldsymbol{B}$ relative to T_B if, and only if, we may conclude on the basis of evidence-based reasoning that it is not the case, for any specified natural number n, that the proposition $\neg F^*(n)$ holds in $\boldsymbol{B}$.

We note that $\boldsymbol{B}$ is a finitary interpretation of PA since—when interpreted suitably—all theorems of first-order PA interpret as *constructively* true in $\boldsymbol{B}$ relative to T_B[10].

This answers question (b).

4. Are *both* interpretations of PA over the structure $\mathbb{N}$ sound?

In the rest of this paper we shall justify that the structure $\mathbb{N}$ can, indeed, be used to define both the standard interpretation $\boldsymbol{M}$, and a finitary interpretation $\boldsymbol{B}$ as above, for PA.

We shall show that, from the PA-provability of $[\neg(\forall x)F(x)]$, we may only conclude under the finitary interpretation $\boldsymbol{B}$, on the basis of evidence-based reasoning, that it is not the case that $[F(n)]$ interprets as always true in $\boldsymbol{B}$ relative to T_B.

We may *not* conclude further on the basis of evidence-based reasoning that $[F(n)]$ interprets as false in $\boldsymbol{B}$ relative to T_B for some numeral $[n]$.

More precisely, we may not conclude from the PA-provability of $[\neg(\forall x)F(x)]$, on the basis of evidence-based reasoning, that the proposition $F^*(n)$ does not hold in $\boldsymbol{B}$ for some natural number n, since we shall show that PA is *not* ω-consistent.

We therefore address the question:

- Are both the interpretations $\boldsymbol{M}$ and $\boldsymbol{B}$ of PA over the structure $\mathbb{N}$ sound, in the sense that the PA axioms interpret as true, and the rules of inference preserve truth, relative to each of the assignments of truth values T_M and T_B respectively?

4.A. Evidence-based reasoning

We begin by noting that the two interpretations $\boldsymbol{M}$ and $\boldsymbol{B}$ of PA over the structure $\mathbb{N}$ can be viewed as complementary, since[11] Tarski's classic definitions permit an intelligence—whether human or mechanistic—to admit *finitary* evidence-based inductive definitions of the satisfaction and truth of the *atomic* formulas of the first-order Peano Arithmetic PA, over the domain N of the natural numbers, in two, hitherto unsuspected and essentially different, ways:

(1) in terms of *classical* algorithmic verifiabilty; and

(2) in terms of *finitary* algorithmic computability.

[9] As detailed in [An12].

[10] Theorem 6.7 in §6.A..

[11] As introduced in [An12] and detailed below.

Thus the PA formula $[(\forall x)F(x)]$, if intended to be read as *'For any x, $F(x)$'*, must be consistently interpreted as:

Definition 3. ***Algorithmic verifiability:***

A number-theoretical relation $F^*(x)$ is algorithmically verifiable if, and only if, for any specified natural number n, there is a deterministic algorithm[12] *$AL_{(F,\ n)}$ which can provide objective evidence for deciding the truth/falsity of each proposition in the finite sequence $\{F^*(1), F^*(2), \ldots, F^*(n)\}$.*

Whereas if $[(\forall x)F(x)]$ is intended to be read as *'For all x, $F(x)$'*, then it must be consistently interpreted as:

Definition 4. ***Algorithmic computability:***

A number theoretical relation $F^*(x)$ is algorithmically computable if, and only if, there is a deterministic algorithm AL_F that can provide objective evidence for deciding the truth/falsity of each proposition in the denumerable sequence $\{F^*(1), F^*(2), \ldots\}$.

We note that algorithmic computability implies the existence of an algorithm that can finitarily decide the truth/falsity of each proposition in a well-defined denumerable sequence of propositions[13], whereas algorithmic verifiability does not imply the existence of an algorithm that can finitarily decide the truth/falsity of each proposition in a well-defined denumerable sequence of propositions.

The following argument shows that although every algorithmically computable relation is algorithmically verifiable, the converse is not true.

Theorem 4.1. *There are number theoretic functions that are algorithmically verifiable but not algorithmically computable.*

Proof: (a) Since any real number R is mathematically definable as the unique limit of a correspondingly unique Cauchy sequence $\{\Sigma_{i=0}^{n} r(i).2^{-i} : n = 0, 1, \ldots; r(i) \in \{0,1\}\}$ of rational numbers in binary notation:

- Where $r(n)$ denotes the n^{th} digit in the decimal expression of the real number $R = Lt_{n\rightarrow\infty}\Sigma_{i=0}^{n} r(i).2^{-i}$ in binary notation.

- Then, for any specified natural number n, Gödel's β-function defines an algorithm $AL_{(R,\ n)}$ that can verify the truth/falsity of each proposition in the finite sequence:

  $\{r(0) = 0, r(1) = 0, \ldots, r(n) = 0\}$.

- Hence, for any real number R, the relation $r(x) = 0$ is algorithmically verifiable trivially.

(b) Since it follows from Alan Turing's Halting argument[14] that there are algorithmically uncomputable real numbers:

- Let $r(n)$ denote the n^{th} digit in the decimal expression of an algorithmically *uncomputable* real number R in binary notation.

[12] A deterministic algorithm computes a mathematical function which has a unique value for any input in its domain, and the algorithm is a process that produces this particularvalue as output.

[13] We note that the concept of 'algorithmic computability' is essentially an expression of the more rigorously defined concept of 'realizability' in [Kl52], p.503.

[14] [Tu36], p.132, §8.

- By (a), the relation $r(x) = 0$ is algorithmically verifiable trivially.

- However, by definition there is no algorithm AL_R that can decide the truth/falsity of each proposition in the denumerable sequence:

  $\{r(0) = 0, r(1) = 0, \ldots\}$.

- Hence the relation $r(x) = 0$ is algorithmically verifiable but not algorithmically computable. □

Since algorithmic verifiability is defined constructively, we note without further comment that the Church-Turing Thesis would not hold if we were to define:

Definition 5. *An arithmetical function is effectively computable if, and only if, it is algorithmically verifiable.*

> **Standard Church's Thesis**[15]: A number-theoretic function (or relation, treated as a Boolean function) is effectively computable if, and only if, it is partial-recursive[16].
>
> **Standard Turing's Thesis**[17]: A number-theoretic function (or relation, treated as a Boolean function) is effectively computable if, and only if, it is Turing-computable[18].

4.B. Reviewing Tarski's inductive assignment of truth-values under an interpretation

We show next that the two definitions correspond to two distinctly different, hitherto unsuspected, assignments of satisfaction and truth to the *compound* formulas of PA over ℕ—T_M and T_B—such that:

- the PA axioms are true over ℕ, and

- the PA rules of inference preserve truth over ℕ,

under both the interpretations $\boldsymbol{M}$ and $\boldsymbol{B}$ relative to T_M and T_B respectively.

We essentially follow standard expositions[19] of Tarski's inductive definitions on the 'satisfiability' and 'truth' of the formulas of a formal language under an interpretation where:

Definition 6. *If* $[A]$ *is an atomic formula* $[A(x_1, x_2, \ldots, x_n)]$[20] *of a formal language S, then the denumerable sequence* $(a_1, a_2, \ldots)$ *in the domain* $\mathbb{D}$ *of an interpretation* $\mathcal{I}_{S(\mathbb{D})}$ *of S satisfies* $[A]$ *if, and only if:*

> *(i)* $[A(x_1, x_2, \ldots, x_n)]$ *interprets under* $\mathcal{I}_{S(\mathbb{D})}$ *as a unique relation* $A^*(x_1, x_2, \ldots, x_n)$ *in* $\mathbb{D}$ *for any witness* $\mathcal{W}_{\mathbb{D}}$ *of* $\mathbb{D}$*;*

---

[15] *Church's (original) Thesis*: The effectively computable number-theoretic functions are the algorithmically computable number-theoretic functions [Ch36].

[16] cf. [Me64], p.227.

[17] After describing what he meant by "computable" numbers in the opening sentence of his seminal paper on Computable Numbers [Tu36], Turing immediately expressed this thesis—albeit informally—as: "... the computable numbers include all numbers which could naturally be regarded as computable".

[18] cf. [BBJ03], p.33.

[19] See Section 12., Appendix A.

[20] We use square brackets to indicate that the contents represent a symbol or a formula of a formal theory, generally assumed to be well-formed unless otherwise indicated by the context.

(ii) there is a Satisfaction Method that provides objective evidence[21] *by which any witness* $\mathcal{W}_{\mathbb{D}}$ *of* $\mathbb{D}$ *can objectively* ***define*** *for any atomic formula* $[A(x_1, x_2, \ldots, x_n)]$ *of S, and any specified denumerable sequence* $(b_1, b_2, \ldots)$ *of* $\mathbb{D}$*, whether the proposition* $A^*(b_1, b_2, \ldots, b_n)$ *holds or not in* $\mathbb{D}$*;*

(iii) $A^*(a_1, a_2, \ldots, a_n)$ *holds in* $\mathbb{D}$ *for any* $\mathcal{W}_{\mathbb{D}}$*.*

Witness: From a constructive perspective, the existence of a 'witness' as in (i) above is implicit in the usual expositions of Tarski's definitions.

Satisfaction Method: From a constructive perspective, the existence of a Satisfaction Method as in (ii) above is also implicit in the usual expositions of Tarski's definitions.

A constructive perspective: We highlight the word '***define***' in (ii) above to emphasise the constructive perspective underlying this paper; which is that the concepts of 'satisfaction' and 'truth' under an interpretation are to be explicitly viewed as objective assignments by a convention that is witness-independent. A Platonist perspective would substitute 'decide' for 'define', thus implicitly suggesting that these concepts can 'exist', in the sense of needing to be discovered by some witness-dependent means—eerily akin to a 'revelation'—if the domain $\mathbb{D}$ is $\mathbb{N}$.

We further define the truth values of 'satisfaction', 'truth', and 'falsity' for the compound formulas of a first-order theory S under the interpretation $\mathcal{I}_{S(\mathbb{D})}$ in terms of *only* the satisfiability of the atomic formulas of S over $\mathbb{D}$ as usual[22].

We then have that[23]:

Theorem 4.2. *(Satisfaction Theorem) If, for any interpretation* $\mathcal{I}_{S(\mathbb{D})}$ *of a first-order theory S, there is an objective Satisfaction Method SM for assigning truth values to the atomic formulas of S, then:*

(i) The Δ_0 *formulas of S are decidable as either true or false (with respect to SM) over* $\mathbb{D}$ *under* $\mathcal{I}_{S(\mathbb{D})}$*;*

(ii) If the Δ_n *formulas of S are decidable as either true or as false over* $\mathbb{D}$ *under* $\mathcal{I}_{S(\mathbb{D})}$*, then so are the* $\Delta(n+1)$ *formulas of S.*

Proof It follows from the above definitions that:

(a) If, for any specified atomic formula $[A(x_1, x_2, \ldots, x_n)]$ of S, it is decidable by $\mathcal{W}_{\mathbb{D}}$ whether or not a sequence $(a_1, a_2, \ldots, a_n)$ of $\mathbb{D}$ satisfies $[A(x_1, x_2, \ldots, x_n)]$ in $\mathbb{D}$ under $\mathcal{I}_{S(\mathbb{D})}$ then, for any specified compound formula $[A^1(x_1, x_2, \ldots, x_n)]$ of S containing any one of the logical constants $\neg, \rightarrow, \forall$, it is decidable by $\mathcal{W}_{\mathbb{D}}$ whether or not the sequence $(a_1, a_2, \ldots, a_n)$ of $\mathbb{D}$ satisfies $[A^1(x_1, x_2, \ldots, x_n)]$ in $\mathbb{D}$ under $\mathcal{I}_{S(\mathbb{D})}$;

(b) If, for any specified compound formula $[B^n(x_1, x_2, \ldots, x_n)]$ of S containing n of the logical constants $\neg, \rightarrow, \forall$, it is decidable by $\mathcal{W}_{\mathbb{D}}$ whether or not a sequence $(a_1, a_2, \ldots, a_n)$ of $\mathbb{D}$ satisfies $[B^n(x_1, x_2, \ldots, x_n)]$ in $\mathbb{D}$ under $\mathcal{I}_{S(\mathbb{D})}$ then, for any specified compound formula $[B^{(n+1)}(x_1, x_2, \ldots, x_n)]$ of S containing $n+1$ of the logical constants $\neg, \rightarrow, \forall$, it is decidable by $\mathcal{W}_{\mathbb{D}}$ whether or not the sequence $(a_1, a_2, \ldots, a_n)$ of $\mathbb{D}$ satisfies $[B^{(n+1)}(x_1, x_2, \ldots, x_n)]$ in $\mathbb{D}$ under $\mathcal{I}_{S(\mathbb{D})}$.

The Theorem follows. □

In other words, if the atomic formulas of of S interpret under $\mathcal{I}_{S(\mathbb{D})}$ as decidable over $\mathbb{D}$ with respect to the Satisfaction Method *SM*, then the propositions of S (i.e., the Π_n and Σ_n formulas of S) also interpret as decidable over $\mathbb{D}$ with respect to *SM*.

We note in particular that:

[21] In the sense of [Mu91].

[22] See §12., Appendix A.

[23] cf. [Me64], pp.51-53.

Theorem 4.3. *A well-formed formula $[F(x)]$ of PA is decidable as true or false under Tarski's truth assignments if, and only if, $[F(x)]$ is algorithmically verifiable.*

Proof The proof follows immediately from Definitions 12 and 13 in §12., since Tarski's definitions are inductive, and a well-formed formula $[F(x)]$ of PA is decidable as true or false under the standard interpretation $\boldsymbol{M}$ of PA over $\mathbb{N}$ if, and only if, each instantiation $[F(n)]$ of $[F(x)]$ is decidable in $\mathbb{N}$. □

We cannot, therefore, assume that the satisfaction and truth of the compound formulas of PA are always finitarily decidable—in the sense of being algorithmically computable—under the standard interpretation $\boldsymbol{M}$ of PA over $\mathbb{N}$, since we cannot prove finitarily from only Tarski's definitions and the assignment T_M of algorithmically verifiable truth values to the atomic formulas of PA under $\boldsymbol{M}$ whether, or not, a quantified PA formula $[(\forall x_i)R]$ is algorithmically verifiable as true under $\boldsymbol{M}$.

We now show how Tarski's definitions yield two distinctly different interpretations of the first-order Peano Arithmetic PA over the domain $\mathbb{N}$ of the natural numbers.

4.C. The ambiguity in the classical standard interpretation of PA over the domain $\mathbb{N}$ of the natural numbers

We note that the classical standard interpretation $\boldsymbol{M}$ of PA over the domain $\mathbb{N}$ of the natural numbers is obtained if, in $\mathcal{I}_{S(\mathbb{D})}$:

(a) we define S as PA with standard first-order predicate calculus as the underlying logic[24];

(b) we define $\mathbb{D}$ as the set $\mathbb{N}$ of natural numbers;

(c) we assume for any atomic formula $[A(x_1, x_2, \ldots, x_n)]$ of PA, and any specified sequence $(b_1^*, b_2^*, \ldots, b_n^*)$ of $\mathbb{N}$, that the proposition $A^*(b_1^*, b_2^*, \ldots, b_n^*)$ is decidable in $\mathbb{N}$;

(d) we define the witness $\mathcal{W}_{\mathbb{N}}$ informally as the 'mathematical intuition' of a human intelligence for whom, classically, (c) has been *implicitly* accepted as *objectively* 'decidable' in $\mathbb{N}$.

(e) we postulate that Aristotle's particularisation holds over $\mathbb{N}$[25].

Clearly, (e) does not form any part of Tarski's *inductive* definitions of the satisfaction, and truth, of the formulas of PA under the above interpretation. Moreover, its inclusion makes $\boldsymbol{M}$ extraneously non-finitary[26].

Moreover, the implicit acceptance in (d) conceals an ambiguity that needs to be made explicit since:

Lemma 4.4. *$A^*(x_1, x_2, \ldots, x_n)$ is both algorithmically verifiable and algorithmically computable in $\mathbb{N}$ by $\mathcal{W}_{\mathbb{N}}$.*

Proof (i) It follows from the argument in Theorem 5.1 (below) that $A^*(x_1, x_2, \ldots, x_n)$ is algorithmically verifiable in $\mathbb{N}$ by $\mathcal{W}_{\mathbb{N}}$.

(ii) It follows from the argument in Theorem 6.1 (below) that $A^*(x_1, x_2, \ldots, x_n)$ is algorithmically computable in $\mathbb{N}$ by $\mathcal{W}_{\mathbb{N}}$. The lemma follows. □

We note without proof that[27] (i) is consistent with, whilst (ii) is inconsistent with, the assumption of Aristotle's particularisation.

[24] Where the string $[(\exists \ldots)]$ is defined as—and is to be treated as an abbreviation for—the PA formula $[\neg(\forall \ldots)\neg]$. We do not consider the case where the underlying logic is Hilbert's formalisation of Aristotle's logic of predicates in terms of his ϵ-operator ([Hi27], pp.465-466).

[25] This postulates that a PA formula such as $[(\exists x)F(x)]$ can always be taken to interpret under $\boldsymbol{M}$ as 'There is some natural number n such that $F(n)$ holds in $\mathbb{N}$.

[26] As argued by Brouwer in [Br08].

[27] For a more detailed argument see [An12].

5. The standard interpretation M of PA

We now argue that:

Definition 7. *An atomic formula* $[A]$ *of PA is satisfiable under the interpretation* $\boldsymbol{M}$ *if, and only if,* $[A]$ *is algorithmically verifiable under* $\boldsymbol{M}$*.*

We note that:

Theorem 5.1. *The atomic formulas of PA are algorithmically verifiable as true or false under the standard interpretation* $\boldsymbol{M}$*.*

Proof It follows from Gödel's definition of the primitive recursive relation xBy[28]—where x is the Gödel number of a proof sequence in PA whose last term is the PA formula with Gödel-number y—that, if $[A(x_1, x_2, \ldots, x_n)]$ is an atomic formula of PA, we can algorithmically verify which one of the instantiations $[A(a_1, a_2, \ldots, a_n)]$ and $[\neg A(a_1, a_2, \ldots, x_a)]$ is necessarily PA-provable and, ipso facto, true under $\boldsymbol{M}$. □

We note that the interpretation $\boldsymbol{M}$ cannot claim to be finitary[29].

Reason: It follows from Theorem 4.1 that we cannot conclude finitarily from Tarski's definitions[30] whether or not a quantified PA formula $[(\forall x_i)R]$ is algorithmically verifiable as true under $\boldsymbol{M}$ if $[R]$ is algorithmically verifiable but not algorithmically computable under the interpretation[31].

5.A. The PA axioms are algorithmically verifiable as true under M

The significance of defining satisfaction in terms of algorithmic verifiability under $\boldsymbol{M}$ is that:

Lemma 5.2. *The PA axioms* PA_1 *to* PA_8[32] *are algorithmically verifiable as true over* $\mathbb{N}$ *under the interpretation* $\boldsymbol{M}$*.*

Proof Since $[x+y]$, $[x\star y]$, $[x = y]$, $[x']$ are defined recursively[33], the PA axioms PA_1 to PA_8[34] interpret as recursive relations that do not involve any quantification. The lemma follows straightforwardly from Theorem 5.1 and Tarski's definitions[35]. □

Lemma 5.3. *For any specified PA formula* $[F(x)]$*, the Induction axiom schema* $[F(0) \rightarrow (((\forall x)(F(x) \rightarrow F(x'))) \rightarrow (\forall x)F(x))]$ *interprets as an algorithmically verifiable true formula under* $\boldsymbol{M}$*.*

Proof

(a) If $[F(0)]$ interprets as an algorithmically verifiable false formula under $\boldsymbol{M}$ the lemma is proved.

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[28] [Go31], p. 22(45).

[29] See [An12] for a proof that $\boldsymbol{M}$ is non-finitary, since it defines a model of PA if, and only if, PA is ω-consistent and so we may always non-finitarily conclude from $[(\exists x)R(x)]$ the existence of some numeral $[n]$ such that $[R(n)]$.

[30] Definition 6 in §4.B., and Definitions 9 to 13 in §12..

[31] Although a proof that such a PA formula exists is not obvious, we shall show that Gödel's 'undecidable' arithmetical formula $[R(x)]$ is algorithmically verifiable but not algorithmically computable under the interpretation $\boldsymbol{M}$.

[32] As detailed in §12..

[33] cf. [Go31], p.17.

[34] As detailed in §12..

[35] Definition 6 in §4.B., and Definitions 9 to 13 in §12..

> *Reason*: Since $[F(0) \rightarrow (((\forall x)(F(x) \rightarrow F(x'))) \rightarrow (\forall x)F(x))]$ interprets as an algorithmically verifiable true formula under $\boldsymbol{M}$ if, and only if, either $[F(0)]$ interprets as an algorithmically verifiable false formula or $[((\forall x)(F(x) \rightarrow F(x'))) \rightarrow (\forall x)F(x)]$ interprets as an algorithmically verifiable true formula under under $\boldsymbol{M}$.

(b) If $[F(0)]$ interprets as an algorithmically verifiable true formula, and $[(\forall x)(F(x) \rightarrow F(x'))]$ interprets as an algorithmically verifiable false formula, under $\boldsymbol{M}$, the lemma is proved.

(c) If $[F(0)]$ and $[(\forall x)(F(x) \rightarrow F(x'))]$ both interpret as algorithmically verifiable true formulas under $\boldsymbol{M}$ then, for any natural number n, there is an algorithm which (by Definition 3) will evidence that $[F(n) \rightarrow F(n')]$ is an algorithmically verifiable true formula under $\boldsymbol{M}$.

(d) Since $[F(0)]$ interprets as an algorithmically verifiable true formula under $\boldsymbol{M}$, it follows for any natural number n that there is an algorithm which will evidence that each of the formulas in the finite sequence $\{[F(0),\ F(1),\ \ldots,\ F(n)\}]$ is an algorithmically verifiable true formula under the interpretation.

(e) Hence $[(\forall x)F(x)]$ is an algorithmically verifiable true formula under $\boldsymbol{M}$.

Since the above cases are exhaustive, the lemma follows. □

We note that if $[F(0)]$ and $[(\forall x)(F(x) \rightarrow F(x'))]$ both interpret as algorithmically verifiable true formulas under $\boldsymbol{M}$, then we can only conclude that, for any natural number n, there is an algorithm which will give evidence for any $m \leq n$ that the formula $[F(m)]$ is true under $\boldsymbol{M}$.

We cannot conclude that there is an algorithm which, for any natural number n, will give evidence that the formula $[F(n)]$ is true under $\boldsymbol{M}$.

Lemma 5.4. *Generalisation preserves algorithmically verifiable truth under* $\boldsymbol{M}$*.*

Proof The two meta-assertions:

'$[F(x)]$ interprets as an algorithmically verifiable true formula under $\boldsymbol{M}$[36]'

and

'$[(\forall x)F(x)]$ interprets as an algorithmically verifiable true formula under $\boldsymbol{M}$'

both mean:

$[F(x)]$ is algorithmically verifiable as true under $\boldsymbol{M}$. □

It is also straightforward to see that:

Lemma 5.5. *Modus Ponens preserves algorithmically verifiable truth under* $\boldsymbol{M}$*.* □

We thus have that:

Theorem 5.6. *The axioms of PA are algorithmically verifiable as true under the interpretation* $\boldsymbol{M}$*, and the rules of inference of PA preserve the properties of algorithmically verifiable satisfaction/truth under* $\boldsymbol{M}$*.* □

[36] See Definition 3

By Theorem 5.1 we conclude that:

Theorem 5.7. *If the the PA-theorems interpret as algorithmically verifiable truths under* $\boldsymbol{M}$*, then PA is consistent.* □

We note that, like Gentzen's argument, such a proof of consistency would be debatably 'finitary', since we cannot conclude from Theorem 5.1 that the quantified formulas of PA are 'finitarily' decidable as true or false under the interpretation $\boldsymbol{M}$.

6. The finitary interpretation $\boldsymbol{B}$ of PA

We next consider a finitary interpretation $\boldsymbol{B}$ of PA, under which we define:

Definition 8. *An atomic formula* $[A]$ *of PA is satisfiable under the interpretation* $\boldsymbol{B}$ *if, and only if,* $[A]$ *is algorithmically computable under* $\boldsymbol{B}$*.*

We note that:

Theorem 6.1. *The atomic formulas of PA are algorithmically computable as true or as false under the finitary interpretation* $\boldsymbol{B}$*.*

Proof If $[A(x_1, x_2, \ldots, x_n)]$ is an atomic formula of PA then, for any specified sequence of numerals $[b_1, b_2, \ldots, b_n]$, the PA formula $[A(b_1, b_2, \ldots, b_n)]$ is an atomic formula of the form $[c = d]$, where $[c]$ and $[d]$ are atomic PA formulas that denote PA numerals. Since $[c]$ and $[d]$ are recursively defined formulas in the language of PA, it follows from a standard result[37] that $[c = d]$ is algorithmically computable as either true or false in $\mathbb{N}$ since there is an algorithm that, for any specified sequence of numerals $[b_1, b_2, \ldots, b_n]$, will give evidence whether $[A(b_1, b_2, \ldots, b_n)]$ interprets as true or false in $\mathbb{N}$. The lemma follows. □

We note that the interpretation $\boldsymbol{B}$ is finitary since:

Lemma 6.2. *The formulas of PA are algorithmically computable finitarily as true or as false under* $\boldsymbol{B}$*.*

Proof The Lemma follows by finite induction from Definition 4, Tarski's definitions[38], and Theorem 6.1. □

6.A. The PA axioms are algorithmically computable as true under $\boldsymbol{B}$

The significance of defining satisfaction in terms of algorithmic computability under $\boldsymbol{B}$ as above is that:

Lemma 6.3. *The PA axioms* PA_1 *to* PA_8[39] *are algorithmically computable as true under the interpretation* $\boldsymbol{B}$*.*

Proof Since $[x+y]$, $[x \star y]$, $[x = y]$, $[x']$ are defined recursively[40], the PA axioms PA_1 to PA_8[41] interpret as recursive relations that do not involve quantification. The lemma follows straightforwardly from Tarski's definitions[42] and Theorem 5.1. □

[37] For any natural numbers m, n, if $m \neq n$, then PA proves $[\neg(m = n)]$ ([Me64], p.110, Proposition 3.6). The converse is obviously true.

[38] Definition 6 in §4.B., and Definitions 9 to 13 in §12..

[39] As detailed in §12..

[40] cf. [Go31], p.17.

[41] As detailed in §12..

[42] Definition 6 in §4.B., and Definitions 9 to 13 in §12..

Lemma 6.4. *For any specified PA formula $[F(x)]$, the Induction axiom schema $[F(0) \rightarrow (((\forall x)(F(x) \rightarrow F(x'))) \rightarrow (\forall x)F(x))]$ interprets as an algorithmically computable true formula under $\boldsymbol{B}$.*

Proof By Tarski's definitions[43]:

(a) If $[F(0)]$ interprets as an algorithmically computable false formula under $\boldsymbol{B}$ the lemma is proved.

> Since $[F(0) \rightarrow (((\forall x)(F(x) \rightarrow F(x'))) \rightarrow (\forall x)F(x))]$ interprets as an algorithmically computable true formula if, and only if, either $[F(0)]$ interprets as an algorithmically computable false formula, or $[((\forall x)(F(x) \rightarrow F(x'))) \rightarrow (\forall x)F(x)]$ interprets as an algorithmically computable true formula, under $\boldsymbol{B}$.

(b) If $[F(0)]$ interprets as an algorithmically computable true formula, and $[(\forall x)(F(x) \rightarrow F(x'))]$ interprets as an algorithmically computable false formula, under $\boldsymbol{B}$, the lemma is proved.

(c) If $[F(0)]$ and $[(\forall x)(F(x) \rightarrow F(x'))]$ both interpret as algorithmically computable true formulas under $\boldsymbol{B}$, then by Definition 4 there is an algorithm which, for any natural number n, will give evidence that the formula $[F(n) \rightarrow F(n')]$ is an algorithmically computable true formula under $\boldsymbol{B}$.

(d) Since $[F(0)]$ interprets as an algorithmically computable true formula under $\boldsymbol{B}$, it follows that there is an algorithm which, for any natural number n, will give evidence that $[F(n)]$ is an algorithmically computable true formula under the interpretation.

(e) Hence $[(\forall x)F(x)]$ is an algorithmically computable true formula under $\boldsymbol{B}$.

Since the above cases are exhaustive, the lemma follows. □

Lemma 6.5. *Generalisation preserves algorithmically computable truth under $\boldsymbol{B}$.*

Proof The two meta-assertions:

'$[F(x)]$ interprets as an algorithmically computable true formula under $\boldsymbol{B}$[44]'

and

'$[(\forall x)F(x)]$ interprets as an algorithmically computable true formula under $\boldsymbol{B}$'

both mean:

$[F(x)]$ is algorithmically computable as true under $\boldsymbol{M}$. □

It is also straightforward to see that:

Lemma 6.6. *Modus Ponens preserves algorithmically computable truth under $\boldsymbol{B}$.* □

We thus have that[45]:

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[43] Definition 6 in §4.B., and Definitions 9 to 13 in §12..

[44] See Definition 4

[45] Without appeal, moreover, to Aristotle's particularisation.

Theorem 6.7. *The axioms of PA are algorithmically computable as true under the interpretation $\boldsymbol{B}$, and the rules of inference of PA preserve the properties of algorithmically computable satisfaction/truth under $\boldsymbol{B}$.* □

We now show that a PA formula $[F(x)]$ is PA-provable if, and only if, $[F(x)]$ is algorithmically computable as true in $\mathbb{N}$. Hence the formulas of PA are finitarily decidable as true/false; whence PA is consistent.

7. Bridging PA Provability and Computability

We note that PA is 'computably' complete in the sense of Hilbert's ω-rule[46], so that:

Theorem 7.1. *(Provability Theorem for PA) A PA formula $[F(x)]$ is PA-provable if, and only if, $[F(x)]$ is algorithmically computable as true in $\mathbb{N}$.*

Proof We have by definition that $[(\forall x)F(x)]$ interprets as true under the interpretation $\boldsymbol{B}$ if, and only if, $[F(x)]$ is algorithmically computable as true in $\mathbb{N}$.

By Theorem 6.7, $\boldsymbol{B}$ defines a finitary model of PA over $\mathbb{N}$ such that:

> If $[(\forall x)F(x)]$ is PA-provable, then $[F(x)]$ is algorithmically computable as true in $\mathbb{N}$;
>
> If $[\neg(\forall x)F(x)]$ is PA-provable, then it is not the case that $[F(x)]$ is algorithmically computable as true in $\mathbb{N}$.

Now, we cannot have that both $[(\forall x)F(x)]$ and $[\neg(\forall x)F(x)]$ are PA-unprovable for some PA formula $[F(x)]$, as this would yield the contradiction:

> (i) There is a finitary model—say $\boldsymbol{B}'$—of PA+$[(\forall x)F(x)]$ in which $[F(x)]$ is algorithmically computable as true in $\mathbb{N}$;
>
> (ii) There is a finitary model—say $\boldsymbol{B}''$—of PA+$[\neg(\forall x)F(x)]$ in which it is not the case that $[F(x)]$ is algorithmically computable as true in $\mathbb{N}$.

The lemma follows. □

Corollary 7.2. *PA is categorical with respect to algorithmic computability.*

Since PA-provability is finitary, the assignment T_B of algorithmically computable truth values to the formulas of PA under $\boldsymbol{B}$ is finitarily decidable.

Hence the PA-theorems interpret as finitary truths under $\boldsymbol{B}$, and we have a finitary proof that:

Theorem 7.3. *PA is consistent.* □

[46]**Hilbert's ω-Rule**: If it is proved that the P-formula $[F(x)]$ interprets as a true numerical formula for each specified P-numeral $[x]$, then the P-formula $[(\forall x)F(x)]$ may be admitted as an initial formula (*axiom*) in P. cf. [Hi30], pp.485-494.

8. Why Hilbert's ε-calculus is not a conservative extension of of the first-order predicate calculus

We further conclude that since Hilbert's ε-calculus admits ε-terms that interpret as 'unspecified' natural numbers, it—contrary to conventional wisdom[47]—is not a conservative extension of of the first-order predicate calculus.

We first note that:

Lemma 8.1. *If $\boldsymbol{M}$ defines the standard model of PA over $\mathbb{N}$, then there is a PA formula $[F]$ which is algorithmically verifiable as true over $\mathbb{N}$ under $\boldsymbol{M}$ even though $[F]$ is not PA-provable.*

Proof Gödel has shown how to construct an arithmetical formula with a single variable—say $[R(x)]$[48]—such that $[R(x)]$ is not PA-provable[49], but $[R(n)]$ is instantiationally PA-provable for any specified PA numeral $[n]$. Hence, for any specified numeral $[n]$, Gödel's primitive recursive relation $xB\lceil[R(n)]\rceil$ must hold for some x. The lemma follows. □

By the argument in Theorem 7.1 it follows that:

Corollary 8.2. *The PA formula $[\neg(\forall x)R(x)]$ defined in Lemma 8.1 is PA-provable.* □

Corollary 8.3. *In any model of PA, Gödel's arithmetical formula $[R(x)]$ interprets as an algorithmically verifiable, but not algorithmically computable, function over $\mathbb{N}$.*

Proof Gödel has shown that $[R(x)]$[50] always interprets as an algorithmically verifiable function over $\mathbb{N}$[51]. By Corollary 8.2 $[R(x)]$ is not algorithmically computable as true in $\mathbb{N}$. □

Corollary 8.4. *PA is not ω-consistent.*[52]

Proof Gödel has shown that if PA is consistent, then $[R(n)]$ is PA-provable for any specified PA numeral $[n]$[53]. By Corollary 8.2 and the definition of ω-consistency, if PA is consistent then it is *not* ω-consistent. □

Theorem 8.5. *A PA formula can denote only algorithmically computable constants.*

Proof Corollary 8.2 implies, under the standard interpretation $\boldsymbol{M}$ of PA, that there is an 'unspecified' natural number q[54] for which the sentence $R^*(q)$ is false.

We thus conclude from Corollary 8.3 that the PA numeral corresponding to such a natural number q is not explicitly definable, by any PA formula, as a first-order term of PA which can be individually denoted within a PA formula. □

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[47] See, for instance, [Sl15].

[48] Gödel refers to this formula only by its Gödel number r ([Go31], p.25(12)).

[49] Gödel's aim in [Go31] was to show that $[(\forall x)R(x)]$ is not P-provable; by Generalisation it follows, however, that $[R(x)]$ is also not P-provable.

[50] Gödel refers to this formula only by its Gödel number r; [Go31], p.25, eqn.12.

[51] [Go31], p.26(2): "$(n)\neg(nB_\kappa(17Gen\ r))$ holds"

[52] This conclusion is contrary to accepted dogma, since ω-consistency (or an equivalent such as Rosser's Rule C) is necessary for concluding the existence of 'undecidable' arithmetical propositions. See, for instance, Davis' remarks in [Da82], p.129(iii) that "...there is no equivocation. Either an adequate arithmetical logic is ω-inconsistent (in which case it is possible to prove false statements within it) or it has an unsolvable decision problem and is subject to the limitations of Gödel's incompleteness theorem".

[53] [Go31], p.26(2).

[54] If we accept the thesis in this related work-in-progress [An14] that—contrary to [Ka91] and [Ka11]—there can be no non-standard numbers in any model of PA.

In other words, a natural number such as q above is an algorithmically uncomputable number, as in the case of the Gödel number of Turing's Halting function.

Whence, it follows from Gödel's reasoning that the PA-numeral corresponding to q is not explicitly definable, by any PA formula, as a first-order term of PA which can be individually denoted within a PA formula[55][56]; even though q is in the domain of the natural numbers that is defined completely by the semantics of Dedekind's second order Peano Postulates[57].

An immediate consequence of this is that Rosser's argument[58] cannot appeal to the introduction of an 'unspecified' PA-numeral—as an instantiation of an existential formula—into a PA-proof sequence by implicitly appealing[59] to the strategem of Rosser's Rule C[60], for concluding the existence of an 'undecidable' Rosser proposition[61] in an arithmetic such as PA.

Corollary 8.6. *Hilbert's ε-calculus is not a conservative extension of the first-order predicate calculus.*

Proof If Hilbert's ε-calculus were a conservative extension of the first-order predicate calculus, then it would be consistent and PA would admit Rosser's proof[62] that the 'Rosser' formula—which is expressed in the language of PA and contains an existential quantifier—is undecidable in the ε-calculus if we define the existential quantifier as in §1.A.IV(13)(1)(ii)[63]. However, by Theorem 7.1, there are no undecidable PA formulas. The corollary follows. □.

Corollary 8.7. *The standard interpretation $\boldsymbol{M}$ of PA does not define a model of PA*[64].

Proof If PA is consistent but not ω-consistent, then Aristotle's particularisation does not hold over $\mathbb{N}$. Since the 'standard' interpretation $\boldsymbol{M}$ of PA appeals to Aristotle's particularisation, the lemma follows. □

9. Why the two interpretations $\boldsymbol{M}$ and $\boldsymbol{B}$ of PA are complementary

The two interpretations $\boldsymbol{M}$ and $\boldsymbol{B}$ of PA can thus be viewed as complementary since:

> (a) if we assume the satisfaction and truth of the *compound* formulas of PA are always *non-finitarily* decidable under $\boldsymbol{M}$, then this assignment corresponds to the classical *non-finitary* standard interpretation $\boldsymbol{M}$ of PA over the domain $\mathbb{N}$ relative to the truth assignments T_M; from which we may further *non-finitarily* conclude on the basis of Gerhard Gentzen's transfinite reasoning that PA is consistent;

[55] See also [Sl15] for a similar, albeit independent, conclusion, based on considerations that can be viewed as a philosophical interpretation of Theorem 8.5.

[56] Philosophically, this would admit the possibility that the behaviour of algorithmically verifiable, but not algorithmically computable, functions may be partially hidden from direct human cognition; a possibility that may have significance for the possible mathematical representation of quantum phenomena in terms of functions that are algorithmically verifiable, but not algorithmically computable, as conjectured in [An13].

[57] [AR02a], p.7, Dedekind's Theorems 132 and 133, and p.3, Definition 3.

[58] [Ro36].

[59] See, for instance, [Me64], p.143, Proposition 3.32.

[60] See [Me64], p.73, §7, Rule C.

[61] Which contains an existentially quantified formula.

[62] [Ro36].

[63] See, for instance, [Me64], p.143, Proposition 3.32.

[64] We note that finitists of all hues—ranging from Brouwer [Br08], to Wittgenstein [Wi78], to Alexander Yessenin-Volpin [He04]—have persistently questioned the assumption that the 'standard' interpretation $\boldsymbol{M}$ can be treated as well-defining a model of PA; see also [Brm07].

whilst:

> (b) the satisfaction and truth of the *compound* formulas of PA are always *finitarily* decidable under the assignment $\boldsymbol{B}$, which corresponds to the *finitary* interpretation $\boldsymbol{B}$ of PA over the domain $\mathbb{N}$ relative to the truth assignments T_B; from which we may further *finitarily* conclude on the basis of evidence-based reasoning that PA is consistent.

We further note that, from such a perspective, the appropriate inference to be drawn from Gödel's 1931 paper[65] is no longer that there exist formally undecidable PA formulas—since PA is *not* ω-consistent—but that we can define PA formulas which, under interpretation, are algorithmically verifiable as true over $\mathbb{N}$, but not algorithmically computable as true over $\mathbb{N}$.

10. Why the two interpretations $\boldsymbol{M}$ and $\boldsymbol{B}$ of PA are *relatively* sound

We conclude that both Hilbert's and Brouwer's interpretations of quantification yield interpretations of the first-order Peano Arithmetic PA—over the structure $\mathbb{N}$ of the natural numbers—that are complementary, not contradictory.

The former yields the standard interpretation $\boldsymbol{M}$ of PA over $\mathbb{N}$, which is sound relative to the assignment T_M of algorithmically verifiable Tarskian truth values to the compound formulas of PA under $\boldsymbol{M}$[66], and which circumscribes the ambit of *non-finitary* human reasoning about 'true' arithmetical propositions.

The latter yields a finitary interpretation $\boldsymbol{B}$ of PA over $\mathbb{N}$, which is sound relative to the assignment T_B of algorithmically computable Tarskian truth values to the compound formulas of PA under $\boldsymbol{B}$[67], and which circumscribes the ambit of *finitary* mechanistic reasoning about 'true' arithmetical propositions.

The complementarity can also be expressed as the thesis that:

Thesis 1. *There can be no mechanist model of human reasoning if the standard interpretation $\boldsymbol{M}$ of PA can be treated as circumscribing the ambit of human reasoning about 'true' arithmetical propositions*[68]*, and the finitary interpretation $\boldsymbol{B}$ of PA can be treated as circumscribing the ambit of mechanistic reasoning about 'true' arithmetical propositions.*

Argument: Gödel has shown how to construct an arithmetical formula with a single variable—say $[R(x)]$[69]—such that $[R(x)]$ is not PA-provable, but $[R(n)]$ is instantiationally PA-provable for any

65[Go31].

66Theorem 5.7 in §5.A.. The soundness follows from Gerhard Gentzen's *non-finitary* proof of consistency for Arithmetic.

67Theorem 6.7 in §6.A.. The soundness follows from the *finitary* proof of consistency for PA detailed therein.

68Such a thesis can be justified by the argument in [An13] that: (i) the standard interpretation $\boldsymbol{M}$ of PA can be viewed as corresponding to the way human intelligence conceptualises, symbolically represents, and logically reasons about, those sensory perceptions that are triggered by physical processes which can be treated as representable—not necessarily *finitarily*—by algorithmically verifiable formulas, where a physical process is effectively computable if, and only if, it is algorithmically verifiable; whilst: (ii) the finitary interpretation $\boldsymbol{B}$ of PA can be viewed as corresponding to the way human intelligence conceptualises, symbolically represents, and logically reasons about, only those sensory perceptions that are triggered by physical processes which can be treated as representable—*finitarily*—by algorithmically computable formulas, where a physical process is effectively computable if, and only if, it is algorithmically computable. We suggest how such a perspective offers a resolution to the EPR paradox.

69Gödel refers to this formula only by its Gödel number r ([Go31], p.25(12)).

specified PA numeral $[n]$. Hence, for any specified numeral $[n]$, Gödel's primitive recursive relation $xB\lceil[R(n)]\rceil$[70] must hold for some natural number m.

If we assume that any mechanical witness can only reason *finitarily* then although, for any specified numeral $[n]$, a mechanical witness can give evidence under the finitary interpretation $\boldsymbol{B}$ that the PA formula $[R(n)]$ holds in $\mathbb{N}$, no mechanical witness can conclude *finitarily* under the finitary interpretation $\boldsymbol{B}$ of PA that, for any specified numeral $[n]$, the PA formula $[R(n)]$ holds in $\mathbb{N}$.

However, if we assume that a human witness can also reason *non-finitarily*, then a human witness *can* conclude under the non-finitary standard interpretation $\boldsymbol{M}$ of PA that, for any specified numeral $[n]$, the PA formula $[R(n)]$ holds in $\mathbb{N}$.

11. The Poincaré-Hilbert debate

We conclude by noting that the complementarity suggested by the preceding perspective appears to resolve the Poincaré-Hilbert debate[71] in Hilbert's favour since:

(i) The axioms of PA are algorithmically verifiable as true under the standard interpretation $\boldsymbol{M}$ of PA, whence Poincaré's argument is invalid if we take it to mean that:

- *Poincaré*: The PA Axiom Schema of Finite Induction cannot be justified relative to algorithmic verifiability under the standard interpretation $\boldsymbol{M}$ of PA, as any such argument would necessarily need to appeal to some form of infinite induction;

(ii) The algorithmically computable finitary interpretation $\boldsymbol{B}$ of PA *validates* Hilbert's belief, if we take this to mean that:

- *Hilbert*: A finitary justification of the PA Axiom Schema of Finite Induction is possible under some finitary interpretation $\boldsymbol{B}$ of PA.

12. Appendix A

Aristotle's particularisation: This holds that from an assertion such as:

'It is not the case that: for any specified x, $P^*(x)$ does not hold',

usually denoted symbolically by '$\neg(\forall x)\neg P^*(x)$', we may always validly infer in the classical, Aristotlean, logic of predicates[72] that:

'There exists an unspecified x such that $P^*(x)$ holds',

usually denoted symbolically by '$(\exists x)P^*(x)$'.

We note that Aristotle's particularisation implies that the classical first-order logic FOL is ω-consistent, and so we may always interpret the formal expression '$[(\exists x)\ F(x)]$' of a formal language under an interpretation as 'There exists an object s in the domain of the interpretation such that $F^*(s)$'.

We note further that Aristotle's particularisation is a *non-finitary* but fundamental tenet of classical logic unrestrictedly adopted as *intuitively obvious* by standard literature[73].

However, L. E. J. Brouwer had noted in his seminal 1908 paper on the unreliability of logical principles[74] that the commonly accepted interpretation of this formula is ambiguous if interpretation is intended over an infinite domain.

Brouwer essentially argued that:

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[70] Where xBy denotes Gödel's primitive recursive relation 'x is the Gödel-number of a proof sequence in PA whose last term is the PA formula with Gödel-number y' ([Go31], p. 22(45)); and $\lceil[R(n)]\rceil$ denotes the Gödel-number of the PA formula $[R(n)]$.

[71] See [Hi27], p.472; also [Br13], p.59; [We27], p.482; [Pa71], p.502-503.

[72] [HA28], pp.58-59.

[73] See [Hi25], p.382; [HA28], p.48; [Sk28], p.515; [Go31], p.32.; [Kl52], p.169; [Ro53], p.90; [BF58], p.46; [Be59], pp.178 & 218; [Su60], p.3; [Wa63], p.314-315; [Qu63], pp.12-13; [Kn63], p.60; [Co66], p.4; [Me64], p.52(ii); [Nv64], p.92; [Li64], p.33; [Sh67], p.13; [Da82], p.xxv; [Rg87], p.xvii; [EC89], p.174; [Mu91]; [Sm92], p.18, Ex.3; [AR02b], p.94, Appendix, Rule 5(i); [BBJ03], p.102; [Cr05], p.6.

[74] [Br08].

(i) Even supposing the formula '$[P(x)]$' of a formal Arithmetical language interprets as an arithmetical relation denoted by '$P^*(x)$'; and

(ii) the formula '$[\neg(\forall x)\neg P(x)]$' interprets as the arithmetical proposition denoted by '$\neg(\forall x)\neg P^*(x)$';

(iii) the formula '$[(\exists x)P(x)]$'—which is formally defined as '$[\neg(\forall x)\neg P^*(x)]$'—need not interpret as the arithmetical proposition denoted by the usual abbreviation '$(\exists x)P^*(x)$'; and

(iv) that such postulation is invalid as a general logical principle in the absence of a means for constructing some putative object a for which the proposition $P^*(a)$ holds in the domain of the interpretation.

Hence we shall follow the convention that the assumption that '$(\exists x)P^*(x)$' is the intended interpretation of the formula '$[(\exists x)P(x)]$'—which is essentially the assumption that Aristotle's particularisation holds over the domain of the interpretation—must always be explicit.

ω-consistency: A formal system S is ω-consistent if, and only if, there is no S-formula $[F(x)]$ for which, first, $[\neg(\forall x)F(x)]$ is S-provable and, second, $[F(a)]$ is S-provable for any specified S-term $[a]$.

In order to avoid intuitionistic objections to his reasoning in his seminal 1931 paper on formally undecidable arithmetical propositions[75], Gödel did not assume that the classical standard assignment $\mathcal{I}_{PA(N,\ S)}$ of PA yields a model of PA. Instead, Gödel introduced the syntactic property of ω-consistency as an explicit assumption in his formal reasoning[76]. Gödel explained at some length[77] that his reasons for introducing ω-consistency as an explicit assumption in his formal reasoning was to avoid appealing to the semantic concept of classical arithmetical truth—a concept which is implicitly based on an intuitionistically objectionable logic that assumes Aristotle's particularisation is valid over $\mathbb{N}$.

However, we note that if we assume the classical standard assignment $\mathcal{I}_{PA(N,\ S)}$ of PA yields a model of PA, then PA is consistent if, and only if, it is ω-consistent. It can thus be argued that Gödel's Platonism was perhaps rooted (justifiably within the context of the implicit *non-finitary* assumption of Aristotle's particularisation in classical theory) in his implicitly held[78] *non-finitary* belief that any first-order axiomatic theory of arithmetic or set theory is ω-consistent.

Standard interpretation of PA: The classical standard interpretation $\boldsymbol{M}$ of PA over the domain $\mathbb{N}$ of the natural numbers is the one in which the logical constants have their 'usual' interpretations[79] in Aristotle's logic of predicates (which subsumes Aristotle's particularisation), and[80]:

(a) The set of non-negative integers is the domain;
(b) The symbol $[0]$ interprets as the integer 0;
(c) The symbol $[']$ interprets as the successor operation (addition of 1);
(d) The symbols $[+]$ and $[\star]$ interpret as ordinary addition and multiplication;
(e) The symbol $[=]$ interprets as the identity relation.

The axioms of first-order Peano Arithmetic (PA)

PA$_1$ $[(x_1 = x_2) \rightarrow ((x_1 = x_3) \rightarrow (x_2 = x_3))]$;
PA$_2$ $[(x_1 = x_2) \rightarrow (x_1' = x_2')]$;
PA$_3$ $[0 \neq x_1']$;
PA$_4$ $[(x_1' = x_2') \rightarrow (x_1 = x_2)]$;
PA$_5$ $[(x_1 + 0) = x_1]$;
PA$_6$ $[(x_1 + x_2') = (x_1 + x_2)']$;
PA$_7$ $[(x_1 \star 0) = 0]$;
PA$_8$ $[(x_1 \star x_2') = ((x_1 \star x_2) + x_1)]$;
PA$_9$ For any well-formed formula $[F(x)]$ of PA:
$[F(0) \rightarrow (((\forall x)(F(x) \rightarrow F(x'))) \rightarrow (\forall x)F(x))]$.

Generalisation in PA If $[A]$ is PA-provable, then so is $[(\forall x)A]$.

Modus Ponens in PA If $[A]$ and $[A \rightarrow B]$ are PA-provable, then so is $[B]$.

Hilbert's Second Problem: "When we are engaged in investigating the foundations of a science, we must set up a system of axioms which contains an exact and complete description of the relations subsisting between the elementary ideas of that science. ... But above all I wish to designate the following as the most important among the numerous questions which can be asked with regard to the axioms: To prove that they are not contradictory, that is, that a definite number of logical steps based upon them can never lead to contradictory results. In geometry, the proof of the compatibility of the axioms can be effected by constructing a suitable field of numbers, such that analogous relations between the numbers of this field correspond to the geometrical axioms. ... On the other hand a direct method is needed for the proof of the compatibility of the arithmetical axioms."[81]

75 [Go31].
76 [Go31], p.23 and p.28.
77 In his introduction on p.9 of [Go31].
78 [Go31], p.28.
79 We essentially follow the definitions in [Me64], p.49.
80 See [Me64], p.107.
81 Excerpted from Maby Winton Newson's English translation [Nw02] of David Hilbert's address [Hi00] at the International Congress of Mathematicians in Paris in 1900.

In this paper, we treat Hilbert's intent[82] behind the enunciation of his Second Problem as essentially seeking a finitary proof for the consistency of arithmetic when formalised in a language such as the first order Peano Arithmetic PA.

Tarski's inductive definitions: We shall assume that truth values of 'satisfaction', 'truth', and 'falsity' are assignable inductively to the compound formulas of a first-order theory S under the interpretation $\mathcal{I}_{S(\mathbb{D})}$ in terms of *only* the satisfiability of the atomic formulas of S over $\mathbb{D}$ as usual[83]:

Definition 9. *A denumerable sequence s of $\mathbb{D}$ satisfies $[\neg A]$ under $\mathcal{I}_{S(\mathbb{D})}$ if, and only if, s does not satisfy $[A]$;*

Definition 10. *A denumerable sequence s of $\mathbb{D}$ satisfies $[A \rightarrow B]$ under $\mathcal{I}_{S(\mathbb{D})}$ if, and only if, either it is not the case that s satisfies $[A]$, or s satisfies $[B]$;*

Definition 11. *A denumerable sequence s of $\mathbb{D}$ satisfies $[(\forall x_i)A]$ under $\mathcal{I}_{S(\mathbb{D})}$ if, and only if, specified any denumerable sequence t of $\mathbb{D}$ which differs from s in at most the i'th component, t satisfies $[A]$;*

Definition 12. *A well-formed formula $[A]$ of $\mathbb{D}$ is true under $\mathcal{I}_{S(\mathbb{D})}$ if, and only if, specified any denumerable sequence t of $\mathbb{D}$, t satisfies $[A]$;*

Definition 13. *A well-formed formula $[A]$ of $\mathbb{D}$ is false under $\mathcal{I}_{S(\mathbb{D})}$ if, and only if, it is not the case that $[A]$ is true under $\mathcal{I}_{S(\mathbb{D})}$.*

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[82] Compare Curtis Franks' thesis in [Fr09] that Hilbert's intent behind the enunciation of his Second Problem was essentially to establish the autonomy of arithmetical truth without appeal to any debatable philosophical considerations.

[83] See [Me64], p.51; [Mu91].

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