Why *Primality* is polynomial time, but *Integer Factorising* is not

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Abstract. We explicate logically different instances that involve defining the probability of an integer n being a prime. We conclude that the prime divisors of a *given* integer are mutually independent; and show why *Primality* is polynomial time, whereas *Integer Factorising* is not.

Keywords. Integer factorising, mutually independent divisors, polynomial-time, primality testing, probability of n being divisible by a prime

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Contents

1. Introduction

We consider the following, logically different, instances that involve defining the probability of an integer[1] n having a property ϕ:

(i) The probability $P_1(n \in \phi)$ of *selecting* an integer n that has the property ϕ from a *given* set S of integers;

(ii) The probability $P_2(n \in \phi)$ that an integer n, in a *given* set S of integers, *has* the property ϕ;

(iii) The probability $P_3(n \in \phi)$ of *determining* that a *given* integer n has the property ϕ.

[1] In what follows we shall use the term 'an integer' to mean 'a natural number', i.e., 'a non negative integer' only.

Taking S as the set N of integers, and ϕ as the property of being a prime, we show that the prime divisors of a *given* integer n are mutually independent[2]; which implies the probability $P_3(n\ is\ a\ prime)$[3] of *determining* that a *given* integer n is prime is $\prod_{i=1}^{\pi(\sqrt{n})}(1-\frac{1}{p_i})$—whence *Primality* is polynomial time—and that the number of primes $\leq \sqrt{n}$ is $O(\frac{\sqrt{n}}{log_e\sqrt{n}})$.

Since determining a factor of n may, in some case, require at least one logical operation for each prime $\leq \sqrt{n}$, we conclude that *Integer Factorising* cannot be done in polynomial time.

2. A 2-dimensional view of Eratosthenes Sieve

We begin by noting how the usual, linearly displayed, Eratosthenes sieve argument exposes the logical structure of divisibility (ipso facto, of primality) when displayed as a 2-dimensional matrix representation of the residues $r_i(n)$, defined for all $n \geq 2$ and all $i \geq 2$ by:

$$n + r_i(n) \equiv 0\ (mod\ i),\ \text{where}\ i > r_i(n) \geq 0.$$

For instance:

> **'Density':** The residues $r_i(n)$ can be defined for all $n \geq 1$ as the values of the non-terminating sequences $R_i(n) = \{i-1,\ i-2,\ \ldots,\ 0,\ i-1,\ i-2,\ \ldots,\ 0,\ \ldots\}$, defined for all $i \geq 1$ (as illustrated in Table 1[4]).

Table 1. The 2-dimensional Eratosthenes 'density' sieve

Sequence:	R_1	R_2	R_3	R_4	R_5	R_6	R_7	R_8	R_9	R_{10}	R_{11}	...	R_n
$n = 1$	0	1	2	3	4	5	6	7	8	9	10	...	n-1
$n = 2$	0	0	1	2	3	4	5	6	7	8	9	...	n-2
$n = 3$	0	1	0	1	2	3	4	5	6	7	8	...	n-3
$n = 4$	0	0	2	0	1	2	3	4	5	6	7	...	n-4
$n = 5$	0	1	1	3	0	1	2	3	4	5	6	...	n-5
$n = 6$	0	0	0	2	4	0	1	2	3	4	5	...	n-6
$n = 7$	0	1	2	1	3	5	0	1	2	3	4	...	n-7
$n = 8$	0	0	1	0	2	4	6	0	1	2	3	...	n-8
$n = 9$	0	1	0	3	1	3	5	7	0	1	2	...	n-9
$n = 10$	0	0	2	2	0	2	4	6	8	0	1	...	n-10
$n = 11$	0	1	1	1	4	1	3	5	7	9	0	...	n-11
n	r_1	r_2	r_3	r_4	r_5	r_6	r_7	r_8	r_9	r_{10}	r_{11}	...	0

We note that:

- For any $i \geq 2$, the non-terminating sequence $R_i(n)$ cycles through the values $(i-1,\ i-2,\ \ldots,\ 0)$ with period i;
- For any $i \geq 2$ the 'density'[5]—over the set of natural numbers—of the set $\{n\}$ of integers that are divisible by i is $\frac{1}{i}$; and the 'density' of integers that are not divisible by i is $\frac{i-1}{i}$.

[2] The conclusion becomes heuristically transparent in §2., which exposes the logical structure of integer divisibility when the usual, linearly displayed, Eratosthenes sieve is displayed as a 2-dimensional matrix.

[3] See §3. (2)(iii).

[4] For r_i read $r_i(n)$; for R_i read $R_i(n)$ in the following table. Note that the values of $R_i(n)$ corresponds to the values of the Bazeries wheel B_i.

[5] Strictly speaking, the probability defined by a discrete probability distribution; see Grinstead and Snell [6], Chapter 5, pp.183-186; also Steuding [5], Chapter 2, p.10.

Moreover:

Primality: The residues $r_i(n)$ can be alternatively defined for all $i \geq 1$ as values of the non-terminating sequences, $E(n) = \{r_i(n) : i \geq 1\}$, defined for all $n \geq 1$ (as illustrated in Table 2).

Table 2. The 2-dimensional Eratosthenes 'primality' sieve

Sequence:	R_1	R_2	R_3	R_4	R_5	R_6	R_7	R_8	R_9	R_{10}	R_{11}	...	R_n
E(1):	*0*	1	2	3	4	5	6	7	8	9	10	...	n-1
E(2):	*0*	*0*	1	2	3	4	5	6	7	8	9	...	n-2
E(3):	*0*	**1**	*0*	1	2	3	4	5	6	7	8	...	n-3
E(4):	*0*	*0*	2	*0*	1	2	3	4	5	6	7	...	n-4
E(5):	*0*	**1**	**1**	**3**	*0*	1	2	3	4	5	6	...	n-5
E(6):	*0*	*0*	*0*	2	4	*0*	1	2	3	4	5	...	n-6
E(7):	*0*	**1**	**2**	**1**	**3**	**5**	*0*	1	2	3	4	...	n-7
E(8):	*0*	*0*	1	*0*	2	4	6	*0*	1	2	3	...	n-8
E(9):	*0*	1	*0*	3	1	3	5	7	*0*	1	2	...	n-9
E(10):	*0*	*0*	2	2	*0*	2	4	6	8	*0*	1	...	n-10
E(11):	*0*	**1**	**1**	**1**	**4**	**1**	**3**	**5**	**7**	**9**	*0*	...	n-11
...													
$E(n)$:	r_1	r_2	r_3	r_4	r_5	r_6	r_7	r_8	r_9	r_{10}	r_{11}	...	*0*

We note that:

- The non-terminating sequences $E(n)$ highlighted in **bold** correspond to a prime p (since $r_i(p) \neq 0$ for any $1 < i < p$) in the usual, linearly displayed, Eratosthenes sieve:

  E(1), **E(2)**, **E(3)**, *E(4)*, **E(5)**, *E(6)*, **E(7)**, *E(8)*, *E(9)*, *E(10)*, **E(11)**, ...

- The non-terminating sequences $E(n)$ highlighted in *italics* identify a crossed out composite n (since $r_i(n) = 0$ for some $1 < i < n$) in the usual, linearly displayed, Eratosthenes sieve.

3. Are the prime divisors of an integer mutually independent?

Assertion: Whether or not a prime p divides an integer n is *not* independent of whether or not a prime $q \neq p$ divides the integer n.

(1) We next note the—not uncommon—*Assertion* that whether or not a prime p divides an integer n is *not* independent of whether or not a prime $q \neq p$ divides the integer n; a belief that may be expressed:

(i) either explicitly:

- "Note that we cannot consider the events $b \equiv 0\ mod(p)$ and $b \equiv 0\ mod(q)$ to be independent, even though p and q are prime, ..." *Regan [1]*.

- "...the probabilities are not independent. ...The probability that a number n is divisible by a prime p is $1/p$, if concerning n we know only that it is large compared with p. If we know that n is *near* N^2 and *not divisible by any prime smaller than* p, then the probability that n is divisible by p is not $1/p$, but f/p." ... *Furry [2].*

• "Prof. E. M. Wright, some months ago, sent me privately a proof on somewhat similar lines that that the probabilities could not be independent for primes greater than $n^{0.76}$." ... *Cherwell [3].*

• "*Find the probability that* x*, a large integer chosen at random, is a prime number.* ... If the integer x is not divisible by any prime p which does not exceed $x^{1/2}$, x itself must be a prime—and so divisibility by primes exceeding $x^{1/2}$ is, in fact, *not* independent of the smaller primes." ... *Pólya [4].*

(ii) or implicitly[6], by arguing that a proof to the contrary must imply that, if $P(n\ is\ a\ prime)$ is the probability that an integer n has the property of being a prime, then $\sum_{i=1}^{\infty} P(i\ is\ a\ prime) = 1$.

(2) We situate the *Assertion* within a more precise perspective, by differentiating between the following probabilities:

(i) The probability $P_1(n \in \phi)$ of *selecting* an integer that has the property ϕ from a *given* set S of integers;

Example 1: If N *is the set of natural numbers, what is the probability of selecting an integer* $n \in N$ *that has the property of being a prime?*

We note that since we cannot define a precise ratio of primes to composites in N, but only an order of magnitude such as $O(\frac{1}{log_e n})$, the probability $P_1(p) \equiv P_1(n \in N\ is\ a\ prime)$ of *selecting* an integer that has the property of being a prime obviously cannot be defined in N.

(ii) The probability $P_2(n \in \phi)$ that an integer, in a *given* set S of integers, *has* the property ϕ;

Example 2: If N^+ *is the set of positive integers, what is the probability that an integer* $n \in N^+$ *is even?*

We note that since any $n \in N^+$ is either odd or even, the probability $P_2(p) \equiv P_2(n \in N^+\ is\ even)$ that an integer $n \in N^+$ *has* the property of being even must be $\frac{1}{2}$. Obviously $P_2(p) \equiv P_2(n \in N^+\ is\ even)$ cannot depend upon the probability $P_1(p) \equiv P_1(n \in N^+\ is\ even)$ of *selecting* an integer $n \in N^+$ that has the property of being even, as the latter would require[7] that $\sum_{i=1}^{\infty} P_2(i \in N^+\ is\ even) = 1$, which is not the case in this example.

[6] cf. *Steuding [5], Chapter 2, p.9, Theorem 2.1.*
[7] Steuding [5], Chapter 2, p.9, Theorem 2.1.

(iii) The probability $P_3(n \in \phi)$ of *determining* that a *given* integer n has the property ϕ.

> *Example 3: I give you a* 5*-digit combination lock along with a* 10*-digit integer* n*. The lock only opens if you set the combination to a proper factor of* n *which is greater than* 1*. What is the probability that the first combination you try will open the lock.*
>
> We note that this is the basis for RSA encryption, which provides the cryptosystem used by many banks for securing their communications.

(3) We shall now show that the unqualified *Assertion* is misleading, by illustrating why the probability $P_3(p|n)$ of *determining* by the spin of a Bazeries wheel whether or not a prime p divides a *given* integer n is $\frac{1}{p}$, and is independent of whether or not a prime $q \neq p$ divides n.

4. Why the prime divisors of an integer are mutually independent

Definition 1. *A modified Bazeries Cylinder*[8] *is a set of polygonal wheels—not necessarily identical (such as* B_i *and* B_j *in Fig. 1 below)—mounted on a common spindle, whose faces are coded with symbols, where the event* $B_i(u)$ *(Fig 2 below) is the value* $0 \leq u \leq i-1$ *yielded by a spin of a single* i*-faced Bazeries wheel* B_i*, and the event* $B_{ij}(u, v)$ *(Fig, 3 below) is the value* (u, v)*—where* $0 \leq u \leq i-1$ *and* $0 \leq v \leq j-1$*—yielded by simultaneous, but independent, spins of an* i*-faced Bazeries wheel* B_i *and a* j*-faced Bazeries wheel* B_j*.* □

Fig. 1. An i-faced Bazeries wheel B_i and a j-faced Bazeries wheel B_j.

Hypothesis 1. *The event yielded by the simultaneous spins of a set of Bazeries wheels is random.* □

(1) We consider first, for any *given* $n > i > 1$, the probability $P_3(B_i(u))$—over the probability space $(0, 1, 2, \ldots, i-1)$—of *determining* that the spin of the Bazeries wheel B_i—with faces numbered $0, 1, 2, \ldots, i-1$—yields the event $B_i(u)$.

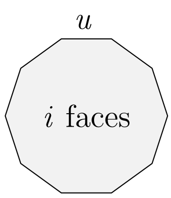

Fig. 2. The event $B_i(u)$ for a single i-faced Bazeries wheel B_i

We conclude by Hypothesis 1 that, for any $0 \leq u \leq i-1$:

[8]Compare *Bazeries cylinder: https://en.wikipedia.org/wiki/Jefferson_disk*.

Lemma 4.1. $P_3(B_i(u)) = \frac{1}{i}$. □

If $n \equiv u \ (mod\ i)$ where $i > u \geq 0$, then i divides n if, and only if, $u = 0$. The probability $P_3(i|n)$ of *determining* by the spin of a Bazeries wheel whether i divides n is thus:

Corollary 4.2. $P_3(i|n) = P_3(B_i(0)) = \frac{1}{i}$. □

Whence the probability $P_3(i \not| n)$ of similarly *determining* that i does not divide n is:

Corollary 4.3. $P_3(i \not| n) = 1 - \frac{1}{i}$. □

(2) We consider next, for any *given* $n > i, \ j > 1$ where $i \neq j$, the compound probability $P_3(B_{ij}(u, v))$ of *determining* whether the simultaneous, but independent, spins of the pair of Bazerian wheels B_i—with faces numbered $0, 1, 2, \ldots, i-1$—and B_j—with faces numbered $0, 1, 2, \ldots, j-1$—yields the event $B_{ij}(u, v)$.

Fig. 3. The event $B_{ij}(u, v)$ for a set of two Bazeries wheels B_i and B_j.

Since the two events $B_i(u)$ and $B_j(v)$ are mutually independent by definition, we conclude by Hypothesis 1 that[9]:

Lemma 4.4. $P_3(B_{ij}(u, v)) = P_3(B_i(u)).P_3(B_j(v)) = \frac{1}{ij}$. □

(3) We conclude further by Hypothesis 1, Lemma 4.1, Corollary 4.2, and Lemma 4.4, that:

Lemma 4.5. $P_3(i|n \ \& \ j|n) = P_3(i|n).P_3(j|n)$ *if, and only if,* $n > i, j > 1$ *and* i, j *are co-prime.* □

Proof. We note that:

(a) The assumption that i, j be co-prime is sufficient. Thus, if i, j are co-prime, and:

$$n \equiv u \ (mod\ i), \ \ n \equiv v \ (mod\ j), \ \ n \equiv w \ (mod\ ij)$$

where $i > u \geq 0, \ j > v \geq 0, \ ij > w \geq 0$, then the ij integers $v.i + u.j$ are all incongruent and form a complete system of residues[10].

Hence $i|n$ and $j|n$ if, and only if, $u = v = 0$.

It follows that $P_3(i|n \ \& \ j|n) = P_3(B_{ij}(0, 0))$.

By Corollary 4.2, $P_3(i|n) = P_3(B_i(0)) = \frac{1}{i}$ and $P_3(j|n) = P_3(B_j(0)) = \frac{1}{j}$.

[9]Grinstead and Snell [6], Chapter 4, §4.1, Definition 4.2, p.141.
[10]Hardy and Wright [7], p.52, Theorem 59.

By Lemma 4.4, $P_3(B_{ij}(0,0)) = \frac{1}{ij}$.

Hence, if i, j are co-prime, then $P_3(i|n \ \& \ j|n) = P_3(i|n).P_3(j|n)$.

(b) The assumption that i, j be co-prime is necessary.

For instance, if $j = 2i$, then $i|n$ and $j|n$ if, and only if, $v = 0$.

Hence $P_3(i|n \ \& \ j|n) = P_3(B_j(0))$

By Corollary 4.2, $P_3(i|n) = P_3(B_i(0)) = \frac{1}{i}$ and $P_3(j|n) = P_3(B_j(0)) = \frac{1}{j}$.

Hence $P_3(i|n \ \& \ j|n) \neq P_3(i|n).P_3(j|n)$.

(4) We thus conclude from Lemma 4.5 that:

Corollary 4.6. *If p and q are two unequal primes, $P_3(p|n \ \& \ q|n) = P_3(p|n).P_3(q|n)$.* □

Theorem 4.7. *The prime divisors of an integer are mutually independent.* □

5. The probability of *determining* that a *given* integer n is a prime

We consider the compound event where $B_i(0)$ does not occur for any of a set of $\pi(\sqrt{n})$ Bazeries wheels.

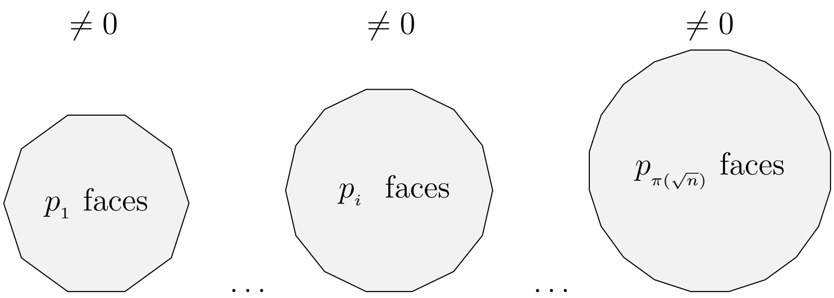

Fig. 4. The event where $B_i(0)$ does not occur for any of a set of $\pi(\sqrt{n})$ Bazeries wheels.

Even though we cannot define the probability $P_1(n \ is \ a \ prime)$ of *selecting* an integer n from the set N of all natural numbers that has the property of being prime[11], since we have by Corollary 4.3 that the probability $P_3(i \not| \ n)$ of *determining* by the spin of a Bazeries wheel that a prime $p < n$ does not divide a *given* n is $1 - \frac{1}{p}$, it follows from Theorem 4.7 that:

Theorem 5.1. *The probability $P_3(n \ is \ a \ prime)$[12] of determining that a given integer n is prime is $\prod_{i=1}^{\pi(\sqrt{n})}(1 - \frac{1}{p_i})$.* □

Proof. By Definition 1, Hypothesis 1, and Lemma 4.4, the probability that $B_i(0)$ does not occur for any i in a simultaneous spin of the $\pi(\sqrt{n})$ Bazeries wheels—where p_i is the i'th prime and B_i has p_i faces (Fig. 4)—is $\prod_{i=1}^{\pi(\sqrt{n})}(1 - \frac{1}{p_i})$.

[11] See §3. (2)(i).
[12] See §3. (2)(iii).

If k is such that $k \not\equiv 0 \ (mod\ p)$ for any prime $p \leq \sqrt{n}$, then the probability $P_3(k\ is\ co-prime\ to\ p \leq \sqrt{n})$ of *determining* by the simultaneous spin of the above $\pi(\sqrt{n})$ Bazeries wheels that k is not divisible by any prime $p \leq \sqrt{n}$ is $\prod_{i=1}^{\pi(\sqrt{n})}(1 - \frac{1}{p_i})$.

In the particular case where n is such that $n \not\equiv 0 \ (mod\ p)$ for any prime $p \leq \sqrt{n}$, the probability $P_3(n\ is\ co-prime\ to\ p \leq \sqrt{n})$ of *determining* by the simultaneous spin of the above $\pi(\sqrt{n})$ Bazeries wheels that n is not divisible by any prime $p \leq \sqrt{n}$ is $\prod_{i=1}^{\pi(\sqrt{n})}(1 - \frac{1}{p_i})$.

Since an integer n is a prime if, and only if, it is not divisible by any prime $p \leq \sqrt{n}$, the theorem follows.

6. Why determining primality is polynomial time $O(log_e n)$

We note the standard definition:

Definition 2. *A deterministic algorithm computes a number-theoretical function $f(n)$ in polynomial-time*[13] *if there exists k such that, for all inputs n, the algorithm computes $f(n)$ in $\leq (log_e\ n)^k + k$ steps.* □

We then have that:

Lemma 6.1. *The minimum number of computational steps needed by a deterministic algorithm*[14] *for determining that a given integer n is prime is $\leq O(log_e n)$.* □

Proof. By Theorem 5.1, the expected number of events which *determine* that a *given* n is prime in a set of k simultaneous spins of the $\pi(\sqrt{n})$ Bazeries wheels—where p_i is the i'th prime and B_i has p_i faces (Fig. 4)—is $k.\prod_{i=1}^{\pi(\sqrt{n})}(1 - \frac{1}{p_i})$; which—by Mertens' Theorem[15] $\prod_{p \leq x}(1 - \frac{1}{p}) \sim \frac{e^{-\lambda}}{log_e x}$—is ≥ 1 if $k \geq \frac{e^{\lambda}}{2}.log_e\ n$. The lemma follows.

By Definition 2, we immediately conclude that:

Theorem 6.2. *Determining whether a given integer n is prime or not can be done in polynomial time $O(log_e n)$.*[16]

7. Why *Integer Factorising* cannot be polynomial-time

Given that n is composite, Theorem 4.7 and Theorem 5.1 now yield the computational complexity consequence that no deterministic algorithm can further compute a factor of n in polynomial time since:

[13] cf. Cook [8], p.1; also Brent [9], p.1, fn.1: "For a polynomial-time algorithm the expected running time should be a polynomial in the length of the input, i.e. $O((logN)^c)$ for some constant c".

[14] A deterministic algorithm computes a mathematical function which has a unique value for any input in its domain, and the algorithm is a process that produces this particular value as output.

[15] Hardy and Wright [7], p. 351, Theorem 22.8; where $\lambda = 0.57722\ldots$ is the Euler-Mascheroni constant and $\frac{e^{\lambda}}{2} = 0.89053\ldots$.

[16] We note that, in a seminal paper 'PRIMES is in P', Agarwal et al [10] have shown that deciding whether an integer n is a prime or not can be done in polynomial time $O(log_e^{15/2} n)$.

Corollary 7.1. *Any deterministic algorithm that always computes a prime factor of n cannot be polynomial-time.* □

Proof. By Theorem 5.1 and Mertens' Theorem, the expected number of primes $\leq \sqrt{n}$ is $O(\frac{\sqrt{n}}{log_e \sqrt{n}})$.

It follows that, if $n = p^k$ for some prime p and $k > 1$, then determining p may require at least one logical operation for algorithmically testing each prime $\leq \sqrt{n}$ deterministically if, for some n, the prime p is the one that is tested last in the particular method of testing the primes $\leq \sqrt{n}$.

Since any algorithmically deterministic method of testing the primes $\leq \sqrt{n}$ must be independent of n, and always have some prime p that is tested last for any *given* n, the algorithm cannot always determine in polynomial time that p is a prime factor of n if $n = p^k$ for some $k > 1$. The theorem follows.

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References

[1] Kenneth W. Regan. (2016, June 29) *Getting to the Roots of Factoring.* In *Gödel's Lost Letter and P = NP* [Web log post]. Retrieved from https://rjlipton.wordpress.com/2016/06/29/getting-to-the-roots-of-factoring/ on 01/08/2016.

[2] W. H. Furry. 1942. *Number of Primes and Probability Considerations*. In *Nature* **150**, 120-121 (25 July 1942), p. 120, doi:10.1038/150120a0

[3] Lord Cherwell. 1942. *Number of Primes and Probability Considerations*. In *Nature* **150**, 120-121 (25 July 1942), p. 121, doi:10.1038/150120a0

[4] G. Pólya. 1959. *Heuristic Reasoning in the Theory of Numbers*. In *The American Mathematical Monthly*, Vol. 66, No. 5 (May, 1959), pp. 375-384, DOI: 10.2307/2308748.

[5] Jörn Steuding. 2002. *Probabilistic Number Theory*. The Pennsylvania State University CiteSeerX Archives, doi=10.1.1.118.4755.

[6] Charles M. Grinstead and J. Laurie Snell. 2003. *Introduction to Probability*, *The CHANCE Project* Version dated 4 July 2006 of the Second Revised Edition, 1997, American Mathematical Society, Rhode Island, USA.

[7] G. H. Hardy and E. M. Wright. 1960. *An Introduction to the Theory of Numbers.* 4th edition. Clarendon Press, Oxford.

[8] Stephen Cook. 2000. *The P versus NP Problem*. Official description provided for the *Clay Mathematical Institute*, Cambridge, Massachusetts.

[9] Richard P. Brent. 2000. *Recent Progress and Prospects for Integer Factorisation Algorithms.* In *Computing and Combinatorics*, Lecture Notes in Computer Science, Volume 1858, 2000, pp.3-22, Springer, New York/Heidelberg.

[10] Manindra Agrawal, Neeraj Kayal, and Nitin Saxena. 2004. *PRIMES is in P*. In *Annals of Mathematics.* Volume 160, Issue 2, pp. 781793.

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